

# Agreement and Disagreement in a Non-Classical World

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## The Classical Agreement Theorem

Alice and Bob possess a common prior probability distribution on a state space

They each then receive different private information about the true state

They form their conditional (posterior) probabilities  $q_A$  and  $q_B$  of an underlying event of interest

*Theorem (Aumann, 1976): If these two values  $q_A$  and  $q_B$  are common knowledge between Alice and Bob, they must be equal*

Here, an event  $E$  is common knowledge between Alice and Bob if they both know it, both know they both know it, and so on indefinitely

# Applications

The agreement theorem is considered a basic requirement in classical epistemics

It has been used to

show that two risk-neutral agents, starting from a common prior, cannot agree to bet with each other

(Sebenius and Geanakoplos, 1983)

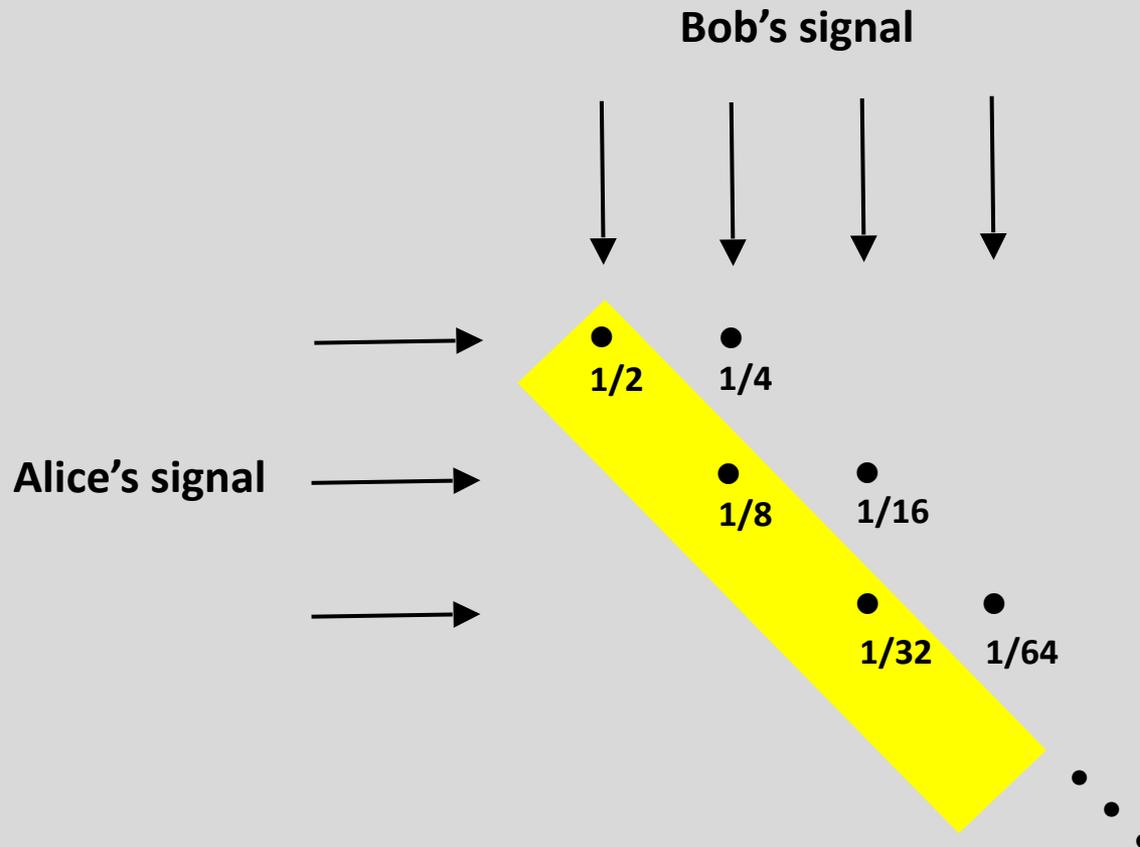
prove “no-trade” theorems for efficient markets

(Milgrom and Stokey, 1982)

establish epistemic conditions for Nash equilibrium

(Aumann and Brandenburger, 1995)

# The Role of Common Knowledge: A “Discontinuity” at Infinity



J. Geanakoplos and H. Polemarchakis, “We Can’t Disagree Forever,” *Journal of Economic Theory*, 28, 1982, 192–200; this variant is due to John Geanakoplos (private communication)

## Non-Classical Settings

What is the status of the agreement theorem when classical probability theory does not apply?

The canonical case is the quantum domain, where a fundamental result (Bell's Theorem, 1964) says that no "local hidden-variable" theory can model the results of all quantum experiments

This implies that the classical Bayesian model does not apply

In this talk, we won't do any physics, but physics will motivate our departure from the Bayesian model

Well, a hint of the physics ...

## 2 x 2 x 2 Boxes

Empirical model:

	(0, 0)	(1, 0)	(0, 1)	(1, 1)
(a, b)	$f_1$	$f_2$	$f_3$	$f_4$
(a', b)	$f_5$	$f_6$	$f_7$	$f_8$
(a, b')	$f_9$	$f_{10}$	$f_{11}$	$f_{12}$
(a', b')	$f_{13}$	$f_{14}$	$f_{15}$	$f_{16}$

Bell model:

	(0, 0)	(1, 0)	(0, 1)	(1, 1)
(a, b)	1/2	0	0	1/2
(a', b)	3/8	1/8	1/8	3/8
(a, b')	3/8	1/8	1/8	3/8
(a', b')	1/8	3/8	3/8	1/8

# Phase Space

	(0, 0)	(1, 0)	(0, 1)	(1, 1)
(a, b)	$f_1$	$f_2$	$f_3$	$f_4$
(a', b)	$f_5$	$f_6$	$f_7$	$f_8$
(a, b')	$f_9$	$f_{10}$	$f_{11}$	$f_{12}$
(a', b')	$f_{13}$	$f_{14}$	$f_{15}$	$f_{16}$

	$a$	$a'$	$b$	$b'$
$p_0$	0	0	0	0
$p_1$	0	0	0	1
$p_2$	0	0	1	0
$p_3$	0	0	1	1
$p_4$	0	1	0	0
$p_5$	0	1	0	1
$p_6$	0	1	1	0
$p_7$	0	1	1	1
$p_8$	1	0	0	0
$p_9$	1	0	0	1
$p_{10}$	1	0	1	0
$p_{11}$	1	0	1	1
$p_{12}$	1	1	0	0
$p_{13}$	1	1	0	1
$p_{14}$	1	1	1	0
$p_{15}$	1	1	1	1

## Signed Probabilities

“Negative energies and probabilities should not be considered as nonsense. They are well-defined concepts mathematically, like a negative of money.”

-- Paul Dirac

Do negative probabilities have uses outside of quantum mechanics?

There is work (e.g., La Mura, 2009) constructing non-classical decision theories with quantum features to resolve the standard paradoxes of choice

Perhaps, there is a more direct route to resolving paradoxes via signed probabilities?

Theorem (Abramsky and Brandenburger, 2011): An empirical model is “no signaling” if and only if there is a phase-space model with a signed probability measure that induces it

## General Set-up

There is a finite abstract state space  $\Omega$

Alice and Bob have partitions  $\mathcal{P}_A$  and  $\mathcal{P}_B$  of  $\Omega$  representing their private information

There is a common – possibly signed – prior probability measure  $p$  on  $\Omega$

Observability: Assume throughout that all members of the partitions  $\mathcal{P}_A$  and  $\mathcal{P}_B$  receive probability in the interval  $(0,1]$ . Assume, too, that all events of interest receive probability in  $(0,1]$ .

## Singular Disagreement – Non-Classical

Alice's (Bob's) partition is red (blue)

The event of interest is

$$E = \{\omega_1, \omega_3, \omega_4\}$$

The true state is  $\omega_1$

At  $\omega_1$ , Alice assigns (conditional) probability 1 to  $E$

At  $\omega_1$ , Bob assigns (conditional) probability 0 to  $E$

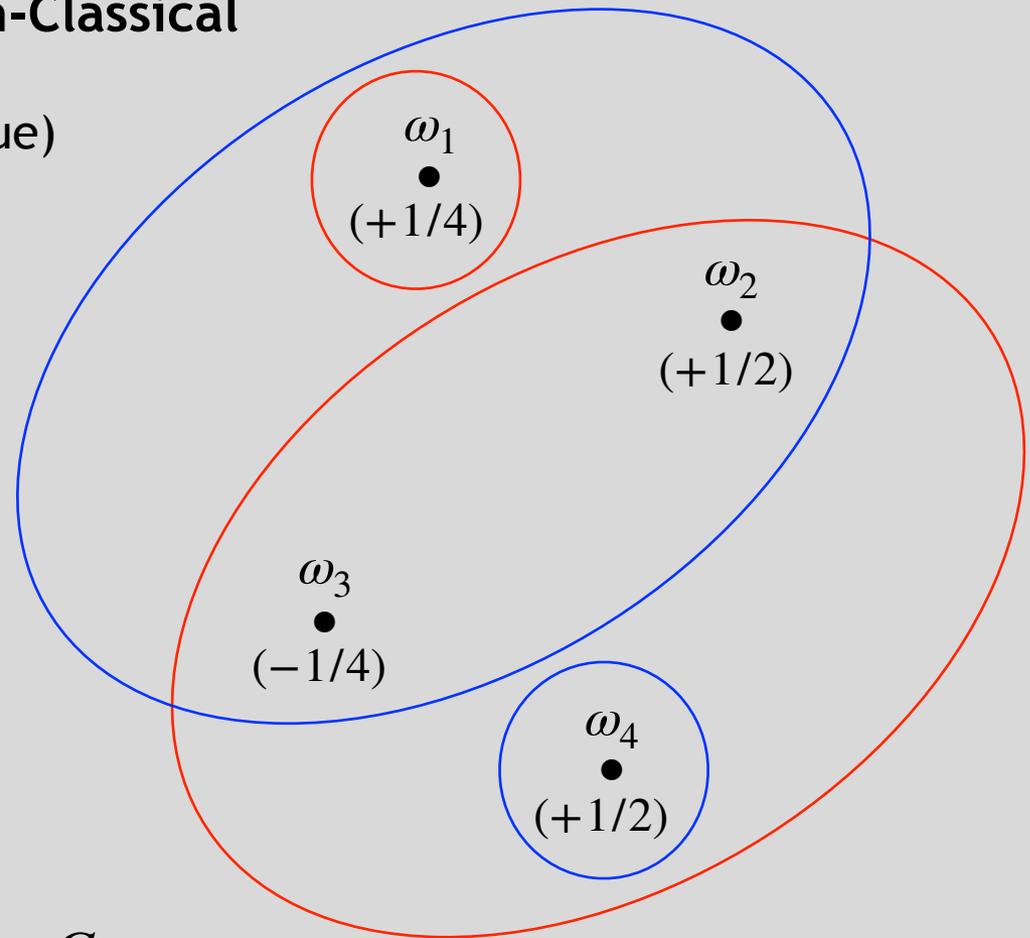
The event that Bob assigns probability 0 to  $E$  is

$$G = \{\omega_1, \omega_2, \omega_3\}$$

At  $\omega_1$ , Alice assigns probability 1 to  $G$

Call this singular disagreement

It is impossible classically!



Note: All partition cells (even events in the join) and  $E$  receive strictly positive probability and are therefore observable

## From Knowledge to Certainty

Definition: Alice *knows* an event  $E$  at state  $\omega$  if  $\mathcal{P}_A(\omega) \subseteq E$

Definition: Alice is *certain of* an event  $E$  at a state  $\omega$  if  $p(E | \mathcal{P}_A(\omega)) = 1$

Fix an event  $E$  and probabilities  $q_A$  and  $q_B$ , and let

$$A_0 = \{ \omega \in \Omega : p(E | \mathcal{P}_A(\omega)) = q_A \}$$

$$B_0 = \{ \omega \in \Omega : p(E | \mathcal{P}_B(\omega)) = q_B \}$$

$$A_{n+1} = A_n \cap \{ \omega \in \Omega : p(B_n | \mathcal{P}_A(\omega)) = 1 \}$$

$$B_{n+1} = B_n \cap \{ \omega \in \Omega : p(A_n | \mathcal{P}_B(\omega)) = 1 \}$$

for all  $n \geq 0$

Definition: It is *common certainty* at a state  $\omega^*$  that Alice assigns probability  $q_A$  to  $E$  and Bob assigns probability  $q_B$  to  $E$  if  $\omega^* \in \bigcap_{n=0}^{\infty} A_n \cap \bigcap_{n=0}^{\infty} B_n$

## Relationship Between Knowledge and Certainty

If Alice knows an event  $E$  at state  $\omega$ , then she is certain of  $E$  at  $\omega$

It is also true that common knowledge of  $E$  implies common certainty of  $E$

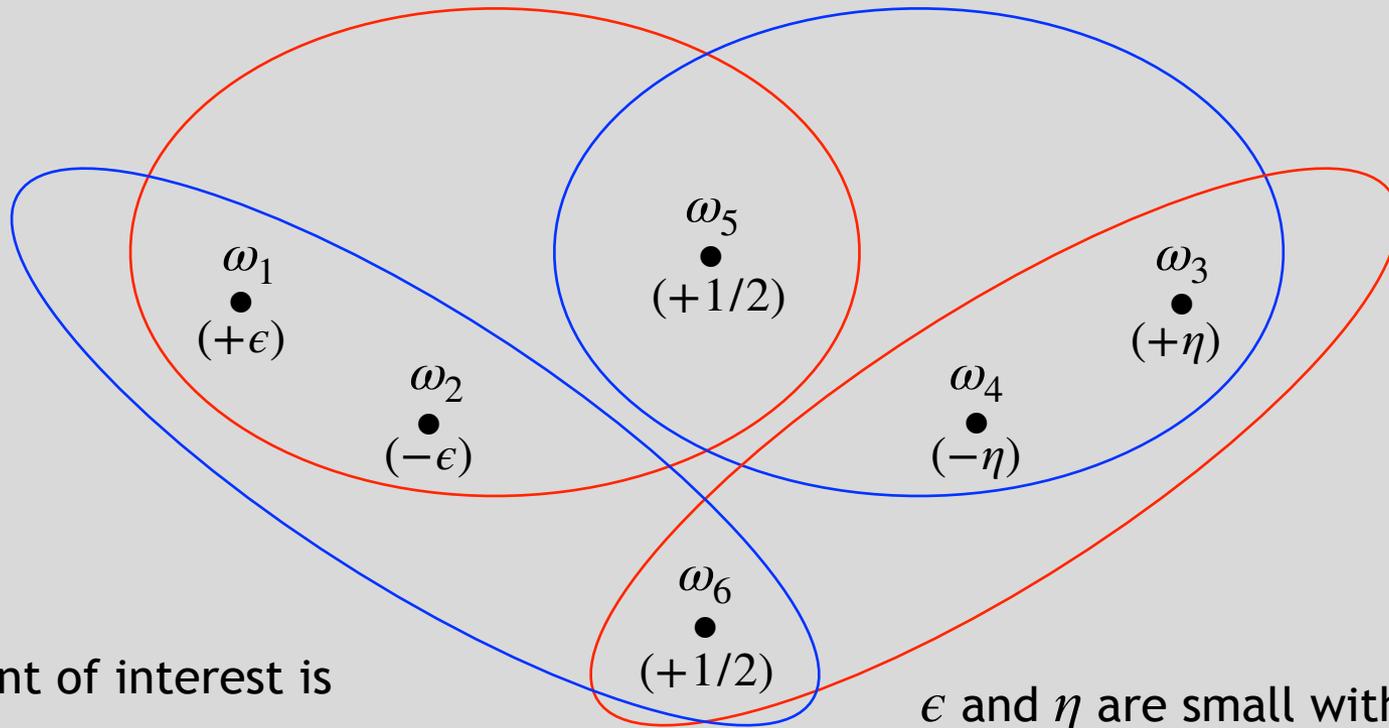
Arguably, the distinction between these modalities is “small” in the classical domain – at least, in the sense that there is also an agreement theorem for common certainty

Theorem (classical): Fix a (non-negative) common prior and an event  $E$ .

Suppose at a state  $\omega^*$  it is common certainty that Alice’s probability of  $E$  is  $q_A$  and Bob’s probability of  $E$  is  $q_B$ . Then  $q_A = q_B$ .

But what happens in the non-classical world?

# Common Certainty of Disagreement



The event of interest is

$$E = \{\omega_2, \omega_4, \omega_5, \omega_6\}$$

$\epsilon$  and  $\eta$  are small with  $\epsilon \neq \eta$

The true state is  $\omega_5$

At  $\omega_5$ , it is common certainty that Alice assigns probability  $1 - 2\epsilon$  to  $E$  while Bob assigns probability  $1 - 2\eta$  to  $E$

That is, there is common certainty of disagreement!

## Realizability of the Counterexample?

Does a scenario of common certainty of disagreement arise in a physical setting?

Physics distinguishes among

- i. classical correlations
- ii. quantum correlations
- iii. postquantum or no-signaling correlations

in  $2 \times 2 \times 2$  boxes as earlier (or in larger boxes)

It turns out that common certainty of disagreement – formulated for boxes – is impossible in domain ii. though possible in domain iii.

Can the impossibility of agreeing to disagree thus be elevated to an (epistemic) physical principle?

## Realizability of the Counterexample Contd.

Could a scenario of common certainty of disagreement arise in a decision-theoretic setting?

Here, we think of the decision makers as equipped with signed probability measures (as suggested earlier)

This would lead to highly non-classical behavior, such as betting between risk-neutral agents ...

Or, should the impossibility of agreeing to disagree be elevated to an (epistemic) decision-theoretic principle?

If yes, what non-classical behavior is then allowed?

## A Note on the Literature

There is a multi-disciplinary literature developing the idea of “quantum-like” decision makers

See e.g. Wang and Busemeyer (2021) for references

Some of this work – see Khrennikov and Basieva (2014) and Khrennikov (2015) – discovers departures from the agreement principle

The relationship between these quantum-based approaches and the signed-measure approach appears to be an open question

Z. Wang and J. Busemeyer, *Cognitive Choice Modeling*, MIT Press, 2021; A. Khrennikov and I. Basieva, “Possibility to Agree on Disagree from Quantum Information and Decision Making,” *Journal of Mathematical Psychology*, 62-63, 2014, 1-15; A. Khrennikov, “Quantum Version of Aumann’s Approach to Common Knowledge: Sufficient Conditions of Impossibility to Agree on Disagree,” *Journal of Mathematical Economics*, 60, 2015, 89-104