Event Valence and Subjective Probability

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Agreement and disagreement in a non-classical world

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The Agreement Theorem Aumann (1976 *Ann. Stat.* **4**, 1236–1239. (doi:10.1214/aos/1176343654)) states that if two Bayesian agents start with a common prior, then they cannot have common knowledge that they hold different posterior probabilities of some underlying event of interest. In short, the two agents cannot 'agree to disagree'. This result applies in the classical domain where classical probability theory applies. But in non-classical domains, such as the quantum world, classical probability theory does not apply. Inspired principally by their use in quantum mechanics, we employ signed probabilities to investigate the epistemics of the non-classical world. We find that here, too, it cannot be common knowledge that two agents assign different probabilities to an event of interest. However, in a non-classical domain, unlike the classical case, it can be common certainty that two agents assign different probabilities to an event of interest. Finally, in a non-classical domain, it cannot be common certainty that two agents assign different probabilities, if communication of their common certainty is possible—even if communication does not take place.

“[T]he theory of probability … has for its main task the study of group phenomena, that is, such phenomena as occur in collections of a large number of objects of essentially the same kind.”

— A. Khinchin and G. Gamow, *Mathematical Foundations of Statistical Mechanics*, 1949

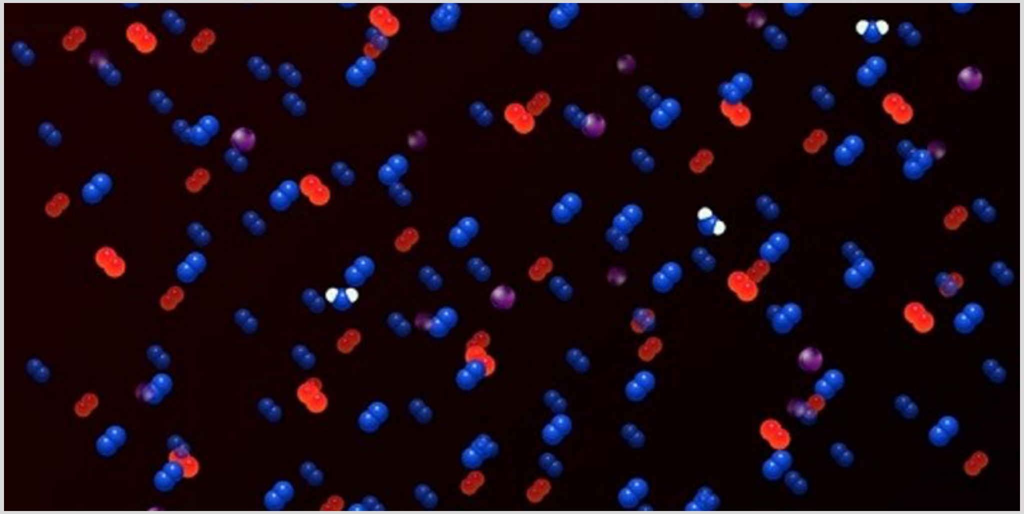


Image from https://www.labroots.com/tag/statistical-mechanics

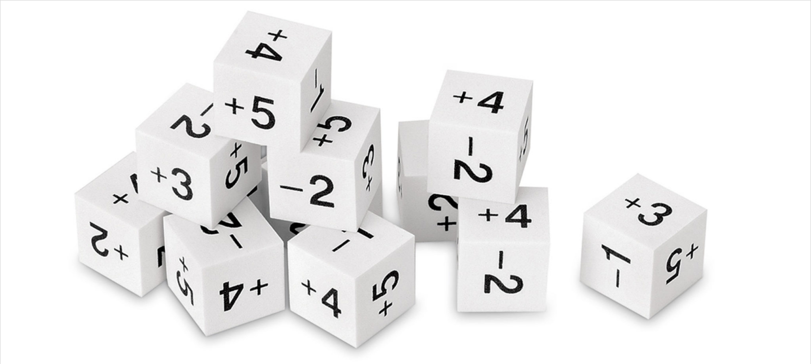
Signed Probabilities

In the domain of subjective probability, a decision-theoretic approach (Ramsey, 1931; de Finetti, 1937; Savage, 1954) views probability as a willingness-to-bet

Why then restrict probabilities to lie in the interval $[0,1]$?

Signed probabilities arise — for a different reason — in the setting of quantum mechanics (Wigner, 1932; Dirac, 1942; Feynman, 1987)

What might a decision theory with signed probabilities contribute here?



Ramsey, F., "Truth and Probability," in Braithwaite, R. (ed.), *The Foundations of Mathematics and other Logical Essays*, Routledge & Kegan Paul, 1931; de Finetti, B., "La Prévision: Ses Lois Logiques, Ses Sources Subjectives, *Annales de l'Institut Henri Poincaré*, 7, 1937, 1-68; Savage, L., *The Foundations of Statistics*, Wiley, 1954; Wigner, E., "On the Quantum Correction for Thermodynamic Equilibrium," *Physical Review*, 1932, 40, 749. 37; Dirac, P., "The Physical Interpretation of Quantum Mechanics," *Proceedings of the Royal Society of London A*, 180, 1942, 621-641; Feynman, R., "Negative Probability," in Hiley, B., and F. Peat (eds.), *Quantum Implications: Essays in Honour of David Bohm*, Routledge & Kegan Paul, 1987, 235-248; image from https://cse.engin.umich.edu/stories/using-negative-probability-for-quantum-solutions

Behavioral Motivation

Consider the event E that your favorite sports team loses the next game

A bet that pays off if E happens would seem to be a "hedge" against disappointment — maybe even traded off against a (small) downside if not-E happens

Instead, in lab-in-field settings, Morewedge et al. (2018), Kossuth et al. (2020), and Donor et al. (2023) found **hedging aversion**

Morewedge, C., S. Tang, and R. Larrick, "Betting Your Favorite to Win: Costly Reluctance to Hedge Desired Outcomes," *Management Science*, 64, 2018, 997-1014; Kossuth, L., N. Powdthavee, D. Harris, and N. Chater, "Does It Pay to Bet on Your Favourite to Win? Evidence on Experienced Utility from the 2018 FIFA World Cup Experiment," *Journal of Economic Behavior & Organization*, 171, 2020, 35-58; Donkor, K., L. Goette, M. Müller, E. Dimant, and M. Kurschilgen, "Identity and Economic Incentives," unpublished, 2023; Illustration 32054918 © Milo827 | Dreamstime.com

Encompassing Other Behavior Effects … and Our Idea

2. The conjunction fallacy (Tversky and Kahneman, 1982, 1983)

3. Co-existence of insurance and betting behavior (Friedman and Savage, 1948)

4. Choice of (FOSD) dominated strategies in strategy-proof mechanisms (Hassidim et al., 2016; Dreyfuss et al., 2022; Shorrer and Sóvágó, 2023)

Our idea:

Events carry a **psychological valence** — which may be attractive (positive) or aversive (negative)

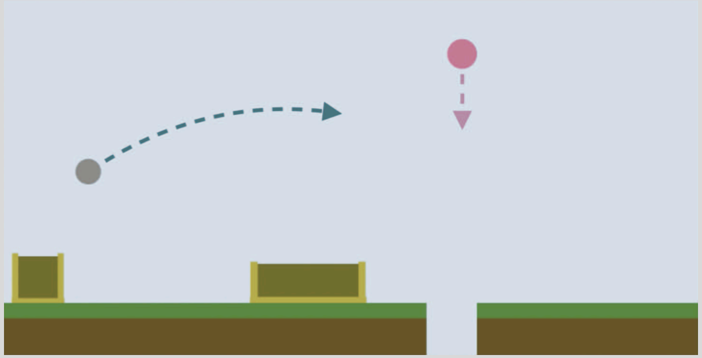
Our philosophy:

Use this one idea to encompass a number of behavioral effects and avoid over-tuning to an individual effect

A consequence may be a reduced fit (e.g., our theory likely works better for small monetary stakes)

Tversky, A., and D. Kahneman, "Judgments of and by Representativeness," In Kahneman, D., P. Slovic, and A. Tversky (eds.), *Judgment under Uncertainty: Heuristics and Biases*, Cambridge University Press, 1982, 84-98; Tversky, A., and D. Kahneman, "Extensional versus Intuitive Reasoning: The Conjunction Fallacy in Probability Judgment," *Psychological Review*, 90, 1983, 293; Friedman, M., and L. Savage, "The Utility Analysis of Choices Involving Risk," *Journal of Political Economy*, 56, 1948, 279-304; Hassidim, A., A. Romm, and R. Shorrer, ""Strategic" Behavior in a Strategy-Proof Environment, in *Proceedings of the 2016 ACM Conference on Economics and Computation*, 2016, 763-764; Dreyfuss, B., O. Heffetz, and M. Rabin, "Expectations-Based Loss Aversion May Help Explain Seemingly Dominated Choices in Strategy-Proof Mechanisms," *American Economic Journal: Microeconomics*, 14, 2022, 515-555; Shorrer, R., and S. Sóvágó, "Dominated Choices in a Strategically Simple College Admissions Environment," *Journal of Political Economy Microeconomics*, 1, 2023, 781-807

Conjunction Reasoning about Physical Events



A pink “sphere” is dropped toward a hole in a grassy field, and a gray “cannonball” travels across the scene in such a way that it could potentially collide with the pink sphere

Question 1 –

How likely is it that the pink sphere will end up on the grass?

Question 2 –

How likely is it that both will happen: The pink sphere will end up on the grass and the cannonball will hit the pink sphere?

Ludwin-Peery, E., et al., “Broken Physics: A Conjunction-Fallacy Effect in Intuitive Physical Reasoning,” *Psychological Science*, 31, 2020, 1602-1611

Review of Anscombe-Aumann Subjective Expected Utility

Let S (finite) be the set of **states of the world**

Let X (a convex subset of a vector space) be the set of **consequences**

A subset E of S is an **event**

A function $f : S \to X$ is an **act**

Let $\mathscr{F}$ denote the set of all acts

A binary relation $\succcurlyeq$ on $\mathscr{F}$ is a **preference relation** for the decision maker (DM)

Anscombe, F., and R. Aumann, "A Definition of Subjective Probability," *Annals of Mathematical Statistics*, 34, 1963, 199-205

Review of SEU: Axioms

Weak Order: The binary relation $\succcurlyeq$ is complete and transitive, and there are $f, g \in \mathscr{F}$ such that $f \succ g$

Independence: If $f, g \in \mathscr{F}$ and $\gamma \in (0,1]$, then $f \succcurlyeq g$ implies $\gamma f + (1-\gamma)h \succcurlyeq \gamma g + (1-\gamma)h$

Archimedean Property: If $f, g \in \mathscr{F}$ and $f \succ g \succ h$, then there are $\alpha, \beta \in (0,1)$ such that $\alpha f + (1-\alpha)h \succ g \succ \beta f + (1-\beta)h$

Monotonicity: For every $f, g \in \mathscr{F}$, if $f(s) \succcurlyeq g(s)$ for every $s \in S$, then $f \succcurlyeq g$

Here, we write $x \succcurlyeq y$ to mean that the constant act that yields x in every state is weakly preferred to the constant act that yields y in every state

Anscombe and Aumann, op.cit.

Non-Monotonicity

States / Acts	Win	Lose
f	0	1
g	0	0

Assume $(1,1) \succcurlyeq (0,0)$ … then Monotonicity precludes $(0,0) \succ (0,1)$

We are looking for a representation:

$$(1-p) \times u(0) + p \times u(1) < u(0)$$

Make p negative!

Making Probability Less Than 0 … and Greater Than 1

Replace Monotonicity with

Indifference Substitution: For every $f, g \in \mathscr{F}$, if $f(s) \sim g(s)$ for every $s \in S$, then $f \sim g$

See Grant and Polak (2013)

Theorem: Weak Order, Independence, Archimedean Property, and Indifference Substitution are necessary and sufficient for there to be a non-constant affine function $u : X \rightarrow \Re$ and a signed probability measure ν on S such that $f \succcurlyeq g$ if and only if

$$\sum_{s \in S} \nu(s) u(f(s)) \geq \sum_{s \in S} \nu(s) u(g(s))$$

Moreover, the function u is cardinally unique and the measure ν is unique

Grant, S., and B. Polak, "Mean-Dispersion Preferences and Constant Absolute Uncertainty Aversion," *Journal of Economic Theory*, 148, 2013, 1361-1398

What Do Signed Probabilities Mean?

A function $\gamma : 2^S \to \mathfrak{R}$ is a **valence** if:

(i) $\gamma(E \cup F) = \gamma(E) + \gamma(F)$ whenever $E \cap F = \varnothing$

(ii) $\gamma(E) + \gamma(E^c) = 0$

Starting from the Jordan decomposition $\nu = \nu^+ - \nu^-$, we can write

$$\nu^+ = (1+b)p^+ \text{ and } \nu^- = bp^-$$

where p^+, p^- are standard probability measures and $b \geq 0$

We call

Likelihood Valence

$$p* = p^+ \text{ and } \gamma* = b(p^+ - p^-)$$

the **minimal decomposition** of the signed probability measure ν

Least Non-Classical Representation

Let ν be a signed probability measure and set

$$P = \{s \in S : \nu(s) > 0\}$$

Theorem: There is a unique decomposition (p^*, γ^*) of ν that satisfies

$$p^* \in \operatorname{argmin}_{q \in \Delta^+(S)} \| \nu - q \|_v$$

(where $\| \cdot \|_v$ is the total variation) and

$$\nu(s)/\nu(s') = p^*(s)/p^*(s') \, \forall \, s, s' \in P$$

In particular

$$\sup_{E \in S} |\gamma^*(E)| \leq \sup_{E \in S} |\gamma(E)|$$

for any alternative decomposition (p, γ)

We call (p^*, γ^*) the **minimal** or **least non-classical** decomposition

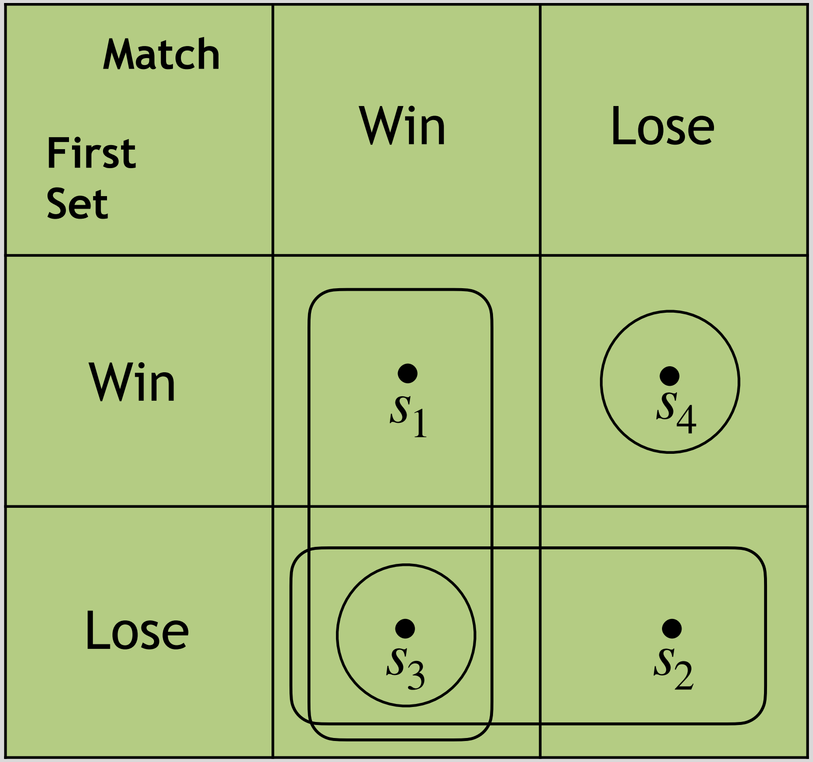
The Conjunction Fallacy

Events

$$E_1 = \{s_1, s_3\},\ E_2 = \{s_2, s_3\}$$
$$E_3 = \{s_3\},\ E_4 = \{s_4\}$$

We are interested in

$$\nu(E_1) > \nu(E_3) > \nu(E_2) > \nu(E_4)$$



Proposition: A sufficient condition for this ranking is that $p(s_1) > -\gamma(s_1)$, $p(s_2) < -\gamma(s_2)$, and $p(s_4) + \gamma(s_4)$ is sufficiently small

Potential Applications to Quantum Theory

In physical systems, probabilities describe frequencies, so that values outside the range $[0,1]$ must be unobservable

Even so, these “hidden” probabilities can affect observable probabilities, as in the phenomenon of **quantum entanglement**

In fact, signed probabilities on phase space characterize the entire physical domain (called the **no-signaling** set) compatible with relativistic causality (Abramsky and Brandenburger, 2011)

> Q: Can these unobserved probabilities be usefully interpreted in the manner of the current paper?
>
> Q: Can our minimal decomposition $\nu = p^* + \gamma^*$ be used as a new measure of non-classicality (called **contextuality**) of physical systems?

Abramsky, S., and A. Brandenburger, “The Sheaf-Theoretic Structure of Non-Locality and Contextuality,” *New Journal of Physics*, 13, 2011, 113036

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