A Co-Creation Value for Cooperative Games

Adam Brandenburger and Barry Nalebuff

Version 03/21/26

The Symmetry Principle

Symmetry plays a central role in the cooperative theory of two-person bargaining games

Consider the potential gain two parties can create over their disagreement outcome

When these gains can be divided up in any fashion — the transferable-utility case — Nash (1950), Shapley (1953), and Kalai and Smorodinsky (1975) all assume it will be divided equally

Under the “divided cloth” rule of O’Neill (1982) and Aumann and Maschler (1985), each claimant first receives the value uncontested by the other, and the remainder is then divided in half

Nash, J., “The Bargaining Problem,” *Econometrica*, 18, 1950, 155-162; Shapley, L., “A Value for n-Person Games,” in Kuhn, H., and A. Tucker (eds.), *Contributions to the Theory of Games II*, Princeton University Press, 1953, 307-317; Kalai, E., and M. Smorodinsky, “Other Solutions to Nash’s Bargaining Problem,” *Econometrica*, 43, 1975, 513-518; O’Neill, B., “A Problem of Rights Arbitration from the Talmud,” *Mathematical Social Sciences*, 2,1982, 345-371; Aumann, R., and M. Maschler, “Game Theoretic Analysis of a Bankruptcy Problem from the Talmud,” *Journal of Economic Theory*, 36, 1985, 195-213

N-Player Games

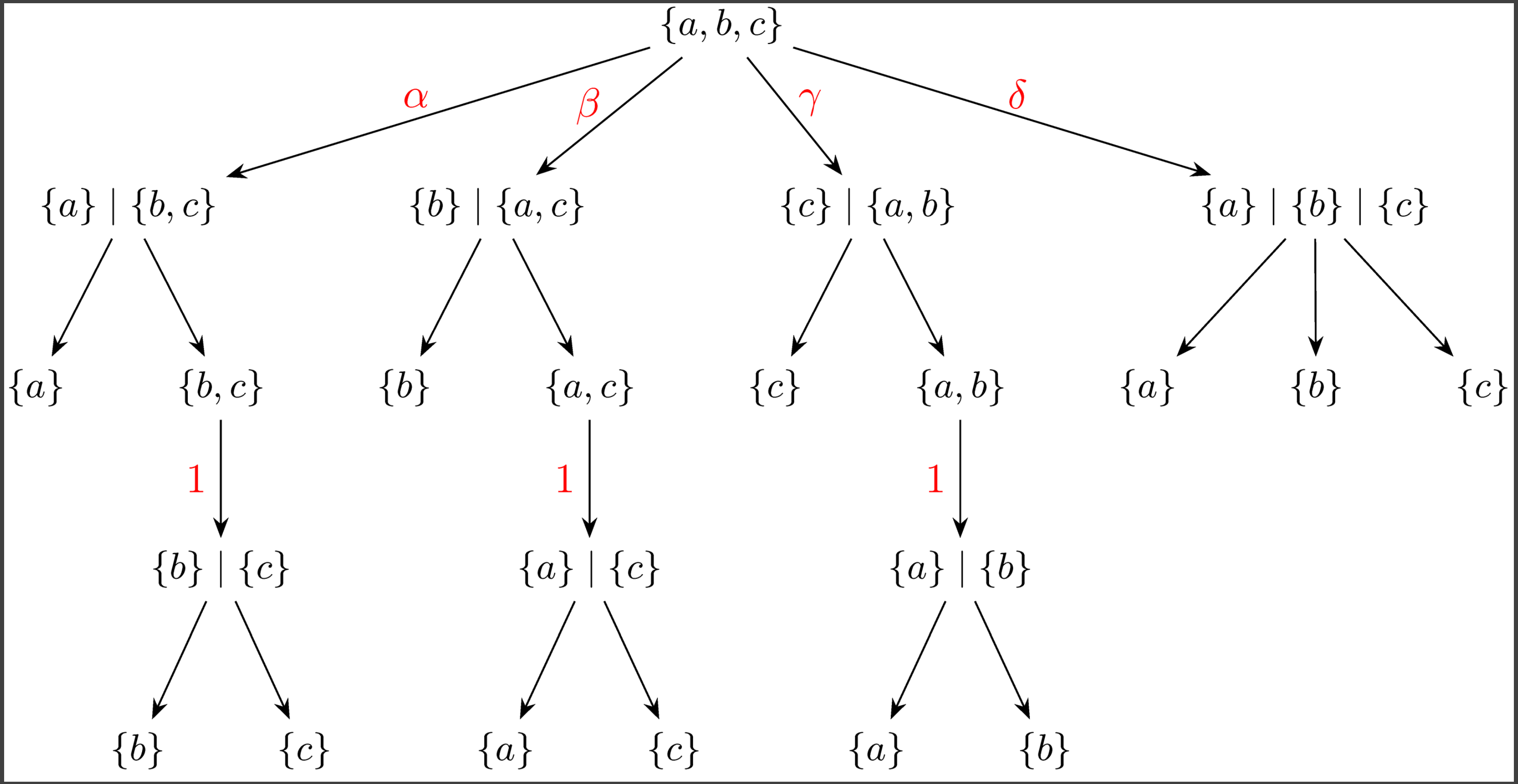
Once there are three or more players, the power of symmetry to determine the solution is greatly diminished

The basic challenge is that the disagreement point is no longer exogenous

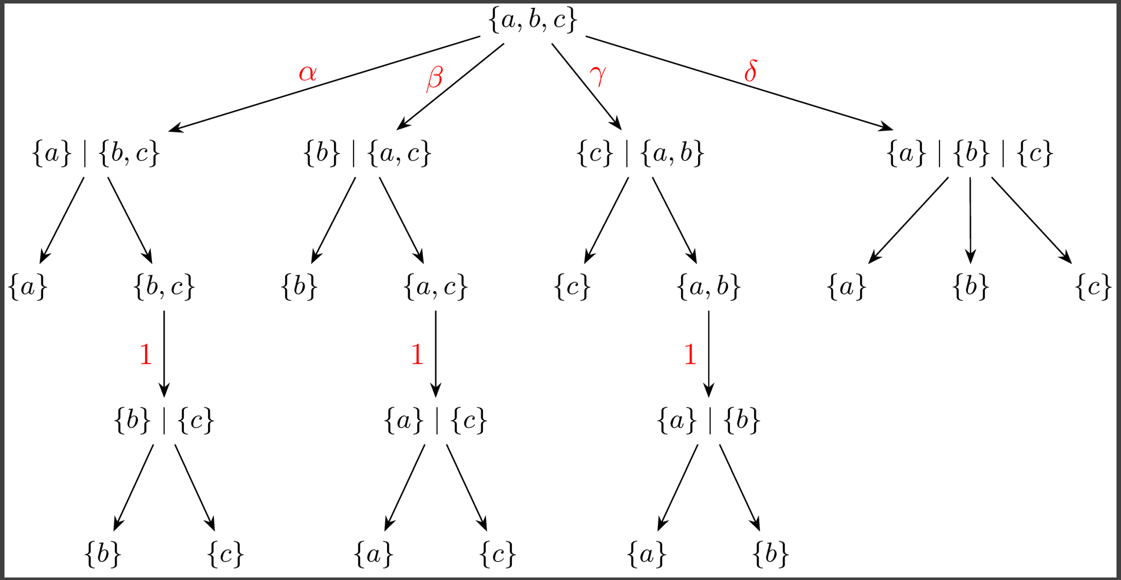
The consequence of non-agreement among three or more players depends on the way in which the multi-player coalition breaks down

There would seem to be more than one way in which this could happen

The Subset-Refinement Game



The CCV Solution Concept



1. The value at each subset is the expected loss on breakdown

2. All players in the subset are needed to reach agreement (symmetry returns)

3. The value at each subset is divided equally

4. Now add for each player the expected (continuation) value after breakdown

5. This defines a solution recursively

Some Initial Properties

1. The expected payoff to player i at subset S is equal to

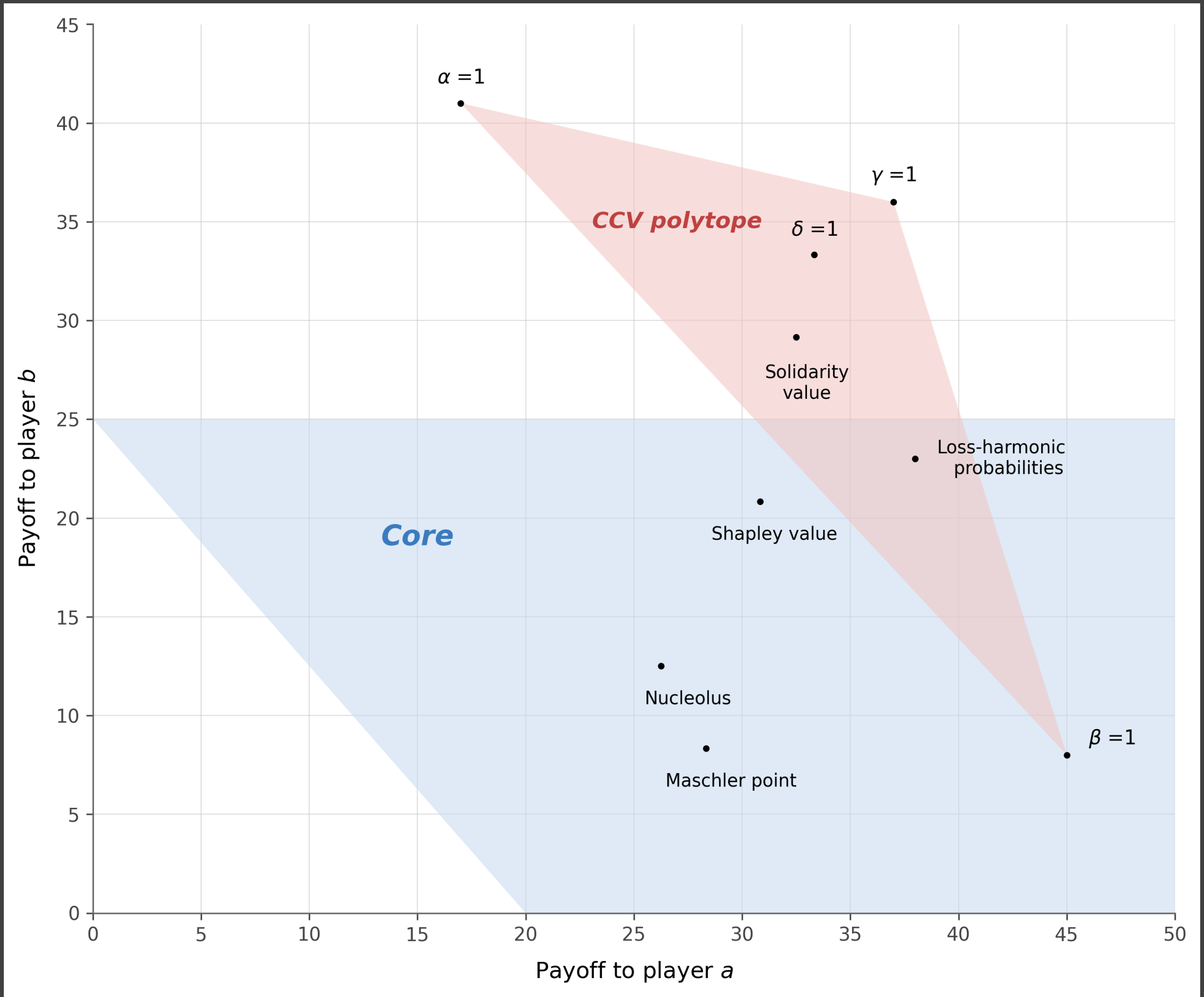
$$\phi_i(S) = \sum_{i \in T \subseteq S} \lambda_S(T) \frac{m(T)}{|T|}$$

where $\lambda_S(T)$ is the hitting probability of T, starting from S, and $m(T)$ is the expected loss at T on breakdown

2. The CCV is affine-invariant

3. For a fixed cooperative game (N, v), the set of CCV expected-payoff vectors, as we vary the family of breakdown probability measures, is a polytope in $\mathbb{R}^{|N|}$

Relationship to Existing Solution Concepts



Maschler, M., "n-Person Games with Only 1, $n-1$, and n-Person Permissible Coalitions," *Journal of Mathematical Analysis and Applications*, 6, 1963, 230-256; Nowak, A., and T. Radzik, "A Solidarity Value For n-Person Transferable-Utility Games," *International Journal of Game Theory*, 23, 1994, 43-48; Schmeidler, D., "The Nucleolus of a Characteristic Function Game," *SIAM Journal on Applied Mathematics*, 6, 1969, 1163-1170; Sprumont, Y., "Population Monotonic Allocation Schemes for Cooperative Games with Transferable Utility," *Games and Economic Behavior*, 2, 1990, 378-394

The Difference from the Shapley Value

Set $\alpha = \beta = \gamma = 1/3$ and $v(\{i\}) = 0$ for each i

Write $l_i = v(\{a, b, c\}) - v(\{j, k\})$ for each top-level local loss

The CCV is

$$\phi_i = 1/9\,(l_a + l_b + l_c) + 1/6\,[v(\{i, j\}) + v(\{i, k\})]$$

The Shapley value is

$$\psi_i = 1/3\,l_i + 1/6\,[v(\{i, j\}) + v(\{i, k\})]$$

The CCV first averages (“collectivizes”) over players the losses under breakdown and then allocates this quantity (equally)

The Shapley value first allocates (“individualizes”) each loss to the respective player and then averages this quantity over orderings (equally)

Applications

Three assumptions on breakdown probabilities:

(i) only partitions where one player is left out receive positive weight

(ii) if the loss when player j is left out is at least as large as when player i is left out, the probability of the first partition is at most equal to the probability of the second partition

(iii) if the loss is 0 when some player i is left out, then only 0-loss partitions receive positive weight and the weights on these partitions are equal

Two examples:

The Glove Game – There are N players with $B = \alpha N$ buyers and $S = (1-\alpha)N$ sellers where $1/2 < \alpha < 1$ and any pair made up of one buyer and one seller creates value of 1

The Overcapacity Game – There are two sellers, each with two units to offer at cost 0, and three buyers, each interested in one unit which they value at 1

Shapley, L., and M. Shubik, "On Market Games," *Journal of Economic Theory*, 1, 1969, 9-25; Aumann, R., "Game Theory," in Durlauf, S., and L. Blume (eds.), *The New Palgrave Dictionary of Economics*, 2nd edition, Palgrave Macmillan, 2008; Postlewaite, A., and R. Rosenthal, "Disadvantageous Syndicates," *Journal of Economic Theory*, 9, 1974, 324-326

The Glove Game

The core gives 0 to each buyer and 1 to each seller

For any $\alpha = 1/2 + \epsilon$, the Shapley value gives close to $1 to each buyer and close to $0 to each seller as $N \to \infty$

The CCV gives $(1 - \alpha)/2\alpha$ to each buyer and $1/2$ to each seller (for any N)

Evidently, the CCV yields a very different solution in this game from either the Shapley value or the core

The Shapley value smooths out the core solution but, unlike the CCV, still exhibits “extreme” behavior in the large-N case

The Overcapacity Game

The core gives 0 to each seller and 1 to each buyer

The Shapley value gives $17/20 = 0.850$ to each seller and $13/30 = 0.433$ to each buyer

The CCV gives $[0.589, 0.617]$ to each seller and $[0.589, 0.607]$ to each buyer

The core appears to underestimate the position of a seller in the game, since, without a seller, the overall value shrinks from 3 to 2

The Shapley value avoids this extreme, but goes very far in the opposite direction in giving sellers nearly double what it gives buyers

The CCV gives a strictly intermediate range of answers and its solution set is closely clustered around an equal five-way division giving $3/5$ to each of the five players

Additional Properties

1. The CCV can be axiomatized via subset-refinement game analogs to Shapley Efficiency, Symmetry, Dummy, and Additivity

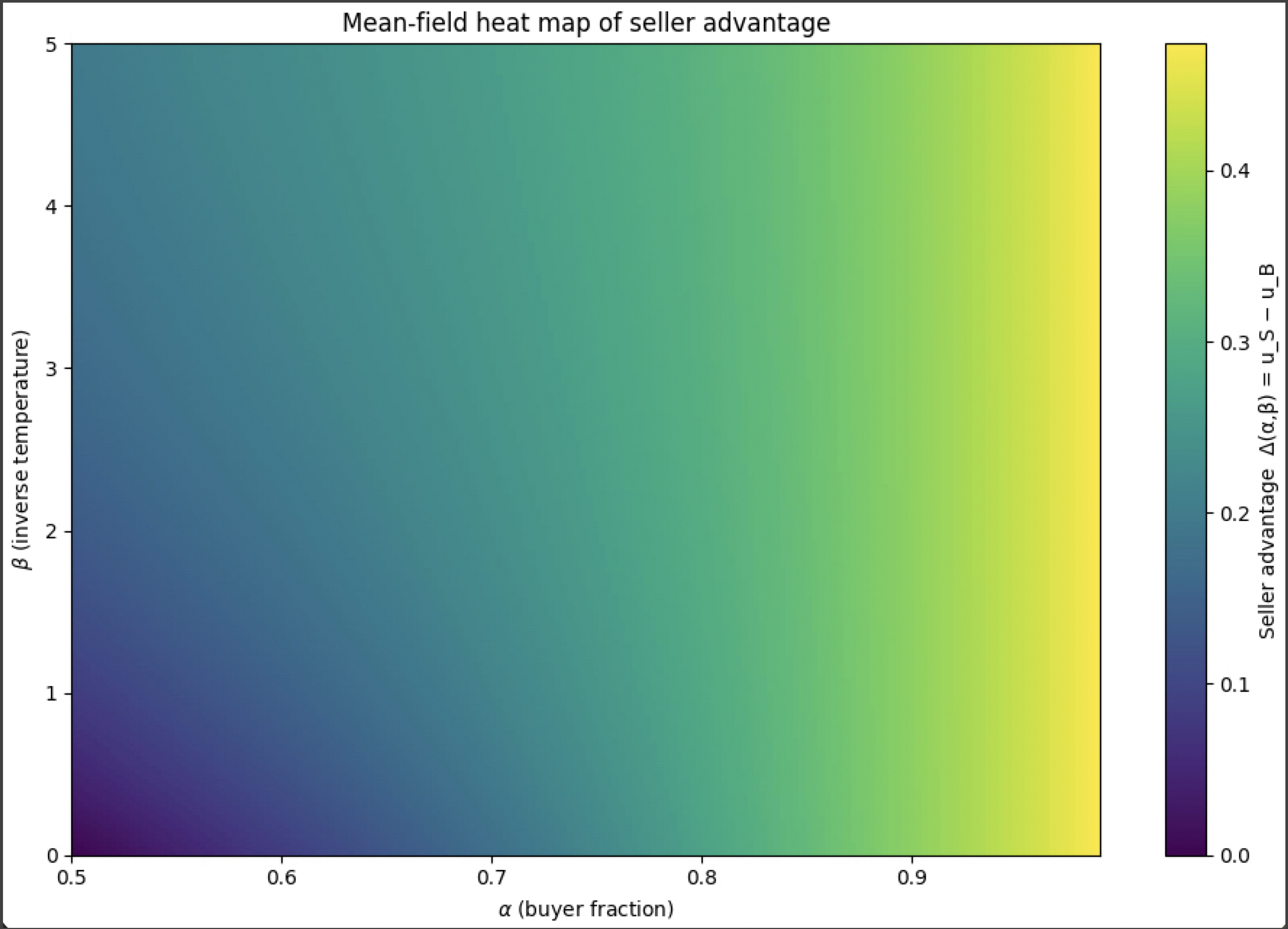
2. The Shapley value can be obtained as the CCV of a game with signed breakdown probabilities

$$p_T(\pi) = (-1)^{x_\pi} (x_\pi - 1)!$$

where x_π is the number of cells in the partition π — and in this case, all hitting probabilities are 1

3. The CCV can be extended to allow context dependency (externalities) by redefining the recursion to operate on pairs (T, σ) where $T \subseteq N$ and σ is a partition-node that produces T in the subset-refinement graph

Myerson, R., "Values of Games in Partition Function Form," *International Journal of Game Theory*, 6, 1977, 23-31


Mean-field heat map of seller advantage
β (inverse temperature)
α (buyer fraction)
Seller advantage Δ(α,β) = u_S − u_B
0
1
2
3
4
5
0.5
0.6
0.7
0.8
0.9
0.0
0.1
0.2
0.3
0.4


Thank You