From Shannon to Signed Rényi:

Entropy When Probabilities Go Negative

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Event Valence

Consider the event E that your favorite sports team loses the next game

A bet that pays off if E happens would seem to be a "hedge" against disappointment — maybe even traded off against a (small) downside if not-E happens

Instead, in lab-in-field settings, Morewedge et al. (2018), Kossuth et al. (2020), and Donor et al. (2023) found "hedging aversion"

Morewedge, C., S. Tang, and R. Larrick, "Betting Your Favorite to Win: Costly Reluctance to Hedge Desired Outcomes," *Management Science*, 64, 2018, 997-1014; Kossuth, L., N. Powdthavee, D. Harris, and N. Chater, "Does It Pay to Bet on Your Favourite to Win? Evidence on Experienced Utility from the 2018 FIFA World Cup Experiment," *Journal of Economic Behavior and Organization*, 171, 2020, 35-58; Donkor, K., L. Goette, M. Müller, E. Dimant, and M. Kurschilgen, "Identity and Economic Incentives," unpublished, 2023; Illustration 32054918 © Milo827 | Dreamstime.com

Negative Probability

States / Acts	Win	Lose
f	0	1
g	0	0

Assume $(1,1) \succcurlyeq (0,0)$ … then Monotonicity precludes $(0,0) \succ (0,1)$

We are looking for a representation:

$$(1-p) \times u(0) + p \times u(1) < u(0)$$

Make p negative!

Brandenburger, A., P. Ghirardato, D. Pennesi, and L. Stanca, "Event Valence and Subjective Probability," 2025, at adambrandenburger.com

What About Entropy?

Jaynes (1957a, 1957b) argued that the "correct" application of the principle of insufficient reason is to choose a probability measure that maximizes entropy subject to the operative constraints

What happens if we blend these two threads?

With negative probabilities, Shannon entropy may involve complex numbers — which would seem to undermine its meaning as the amount of uncertainty in a system

Negative probabilities also necessarily arise in phase-space representations of quantum mechanics (but this is another story)

Define signed Shannon entropy as a fix?

$$H^{\pm}(p) = -\sum_i |p_i| \log |p_i|$$

Jaynes, E., "Information Theory and Statistical Mechanics," *The Physical Review*, 106, 1957a, 620—630;
Jaynes, E., "Information Theory and Statistical Mechanics II," *The Physical Review*, 106, 1957b, 171-190

From Unsigned to Signed Rényi Entropy

Bookkeeping axioms together with extensivity

$$H(p \otimes q) = H(p) + H(q)$$

characterize the Rényi parametric family (for $\alpha > 0$ with $\alpha \neq 1$)

$$H_\alpha(p) = -\frac{1}{\alpha - 1} \log\Big(\frac{\sum_i p_i^\alpha}{\sum_i p_i}\Big)$$

Requiring real-valuedness and extending continuity to negative values characterizes:

$$H_\alpha^\pm(p) = -\frac{1}{\alpha - 1} \log\Big(\frac{\sum_i |p_i|^\alpha}{|\sum_i p_i|}\Big)$$

Rényi, A., "On Measures of Information and Entropy," in Neyman, J. (ed.), *Proceedings of the 4th Berkeley Symposium on Mathematical Statistics and Probability*, University of California Press, 1961, 547-561; Brandenburger, A., and P. La Mura, "Signed Rényi Entropy and Quantum Second Laws," 2024, at https://arxiv.org/abs/2410.15976

Properties of Signed Rényi Entropy

Note that signed Rényi entropy generally diverges at $\alpha = 1$ and does not yield signed Shannon entropy there

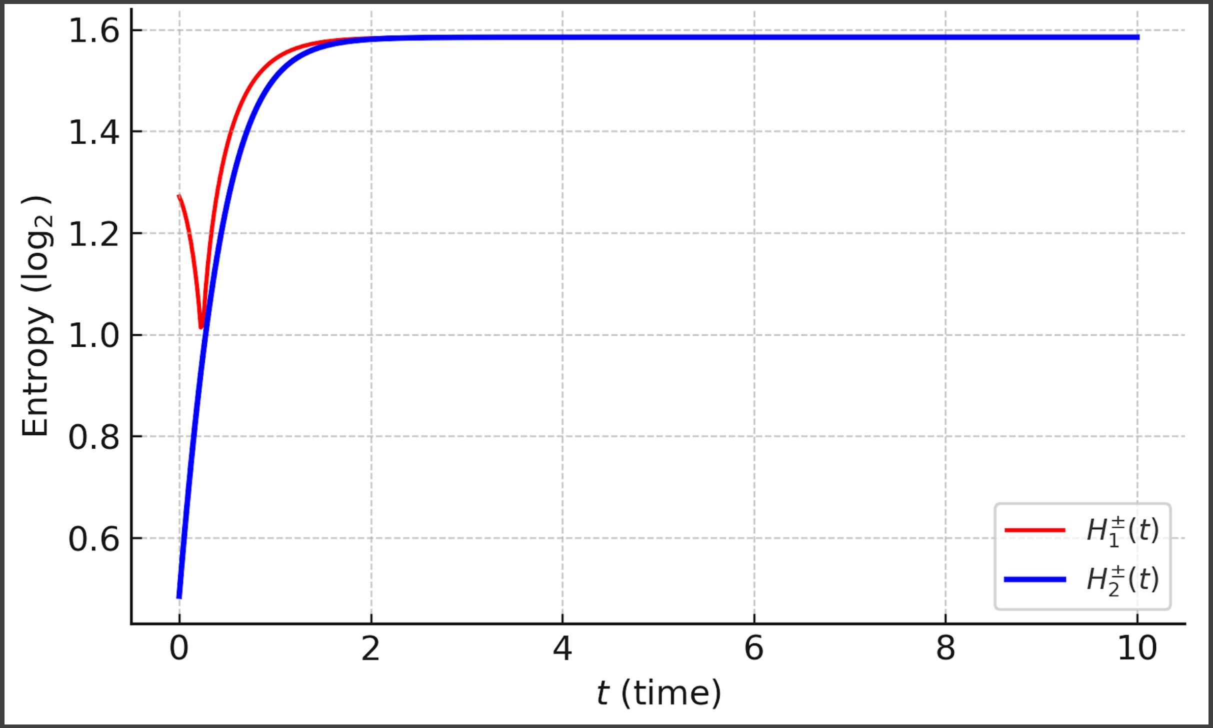
Theorem: *A signed (probability) measure* p *contains at least one strictly negative entry if and only if there is an* $\alpha > 1$ *such that* $H_\alpha^{\pm}(p) < 0.$ *Signed Shannon entropy does not witness negativity in this way.*

Theorem: *If* $\alpha > 1,$ *then* $H_\alpha^{\pm}(p)$ *is Schur-concave. Signed Shannon entropy is not Schur-concave.*

Brandenburger, A., and P. La Mura, op.cit.; Koukoulekidis, N., and D. Jennings, “Constraints on Magic State Protocols from the Statistical Mechanics of Wigner Negativity,” *npj Quantum Information*, 8, 2022, 42; Ruodu Wang generously shared a significant extension of the second theorem

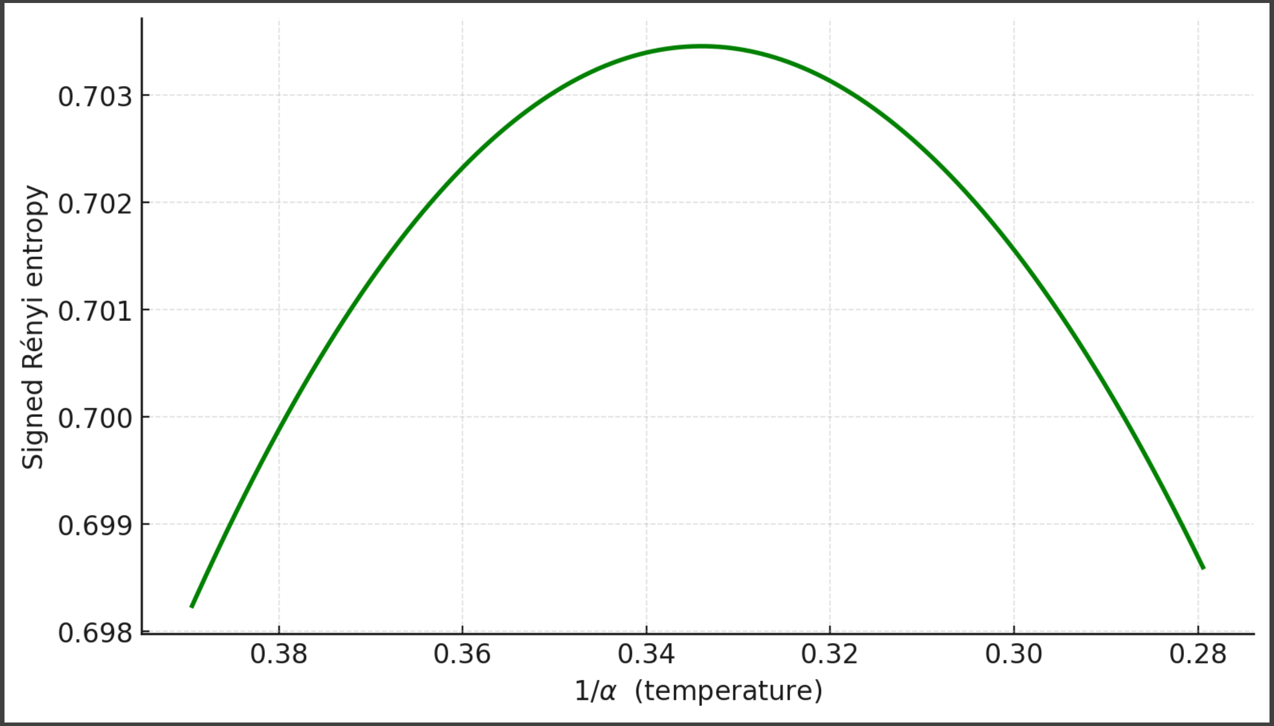
Properties of Signed Rényi Entropy contd.

Theorem (a signed H-Theorem): *Let* p *evolve according to the Markov process* $d/dt\,\vec{p}(t) = \Lambda\vec{p}(t)$ *where* Λ *is a negative Laplacian. If* $\alpha > 1$, *then* $d/dt\,H_\alpha^{\pm}(p(t)) \geq 0$.



Properties of Signed Rényi Entropy contd.

For unsigned probability measures, Rényi entropy is non-increasing in α. Signed Rényi entropy can be non-monotonic. This has physical implications when α is interpreted as a (dimensionless) inverse temperature parameter. There may also be interesting decision-theoretic implications.



Brandenburger, A., and P. La Mura, op.cit.


Thank You