A Co-Creation Value for Cooperative Games*

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Abstract

We introduce a new solution concept for cooperative games that starts by augmenting the traditional description to include probabilities specifying how each coalition would break down into sub-coalitions in the absence of agreement. The resulting "no-agreement" outcome is the baseline that determines the expected gain from cooperation. Since agreement among all players in a coalition is needed to avoid breakdown, we propose this gain be shared equally among them. Recursive application of this principle on a subset-refinement graph leads to our Co-Creation Value (CCV). The CCV incorporates a different philosophy from the Shapley value. While the latter effectively allocates a loss on breakdown to the player responsible and then averages over orderings of players, the CCV treats loss as a collective phenomenon—we first average losses and then allocate this average equally. In short, we take a collective vs. individualistic view of the gains and losses in a game. Nevertheless, we can recover the Shapley value as a special case of the CCV under unit hitting probabilities of all subsets. We further show how the Shapley axioms can be modified to obtain our solution concept. For convex games, we prove that the CCV polytope has a nonempty intersection with the core. We illustrate the CCV for a series of economic scenarios modeled as cooperative games. Extensions to our basic framework include a detailed treatment of externalities, an extension to NTU games, and derivations via consistency and monotonicity axioms.

1 Introduction

Symmetry plays a central role in the cooperative theory of two-person bargaining games. Consider the potential gain two parties can create over their disagreement outcome. When these gains can be divided up in any fashion—the transferable-utility case—Nash (34, 1950), Shapley (45, 1953), and Kalai and Smorodinsky (16, 1975) all assume it will be divided equally. Under the "divided cloth" rule of O'Neill (37, 1982) and Aumann and Maschler (3, 1985), each claimant first receives the value uncontested by the other, and the remainder is then divided in half.

Once there are three or more players in a bargaining game, the power of symmetry to determine the solution is greatly diminished and different solution concepts have proliferated over the years. (Moulin 32, 2004, and Owen 38, 2013 are standard references.) The basic challenge is that the disagreement point is no longer exogenous. The consequence of non-agreement among three or more players depends on the way in which the multi-player coalition breaks down—and there is generally more than one way in which this could happen. If we can arrive at a natural analysis of this question, the potential to apply a symmetry principle at the level of a multi-player coalition returns.

In this paper, we augment the traditional definition of a cooperative game to include a probability graph describing how each coalition of players will break down in the absence of agreement. With these breakdown probabilities, there is a well-defined expected gain to associate with each coalition, calculated by taking the expectation of the difference between the value of the coalition and the total value of each partition of that coalition in the event of a breakdown. Since agreement among all players in a coalition is needed to avoid breakdown, we appeal to symmetry to divide this expected gain equally among coalition members. This leads to a recursively defined overall value of the game to each player. We call our solution concept the co-creation value (CCV) to emphasize the equal importance of all players in reaching agreement.

A second way to motivate our solution concept is to start with the n-unanimity game (Shapley 45, 1953), which is the game where only the grand coalition creates nonzero value. Conceptually, all players in this game have equal power, and the obvious solution is to divide the value of the grand coalition equally. Any

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other solution would involve an asymmetry not present in the description of the game. The CCV solution extends this logic to each subset of players, where the value is the expected gain relative to breakdown. Analogous to the case of the grand coalition in a unanimity game, all players in the subset in question are needed for this expected gain to be realized. Seen this way, our restoration of a symmetry principle to games with three or more players is akin to the classic unanimity analysis—only now applied recursively within a game.

Naturally, there are important connections between our solution and existing ones. In the Shapley value, a player who joins a coalition is given its entire marginal contribution. Our perspective is that the existing players in a coalition are equally required for this contribution to arise. This idea of dividing the marginal contributions equally among all relevant players was pioneered by Sprumont (52, 1990) and axiomatized by Nowak and Radzik (36, 1994) in their Solidarity value. We depart from this specific resolution because we take the expected gain from an agreement to form a coalition to be calculated relative to all possible breakdowns, and these breakdowns need not have equal probabilities. The Harsanyi (12, 1959) dividend approach converts a game into symmetric component games, whose symmetric solutions sum to the solution of the overall game. This approach relies on a specific form of additivity and leads back to the Shapley value. We also distinguish our solution concept from the nucleolus (Schmeidler 44, 1969) and the Maschler (25, 1963) solution.

A different connection to the literature arises from considering cooperative games with externalities, represented in partition-function form (Lucas 21, 1963; Thrall and Lucas 55, 1963). Our recursively-defined solution concept extends naturally to encompass externalities, by allowing the value of the characteristic function at a given subset, and the breakdown probabilities, there, to depend on the context—more formally, on the path in the graph to that subset. We illustrate this extended framework and then use it to examine the long-standing question of what should be the extension of the Shapley value to games with externalities (Myerson, 33, 1977; Bolger, 5, 1989; Macho-Stadler, Pérez-Castrillo, and Wettstein, 23, 2007; de Clippel and Serrano, 8, 2008; McQuillin, 30, 2009; Dutta, Ehlers, and Kar, 10, 2010; Skibski and Michalak 51, 2020; and Navarro, Rémila, and Solal 35, 2025). In a particular example of a game with externalities (de Clippel and Serrano, 8, 2008), we are able to derive these different approaches within our one framework, which we think helps to make transparent the different assumptions underneath them.

We go deeper into the relationship between the CCV and the Shapley value (in the base case of no externalities) by axiomatizing our solution concept via modifications of the Shapley axioms of Efficiency, Symmetry, Dummy, and Additivity (Shapley 45, 1953). Efficiency and Symmetry carry over directly. Our definition of a dummy player is importantly different. In our framework, a player is a dummy if breakdowns are costless whenever the player is present. In particular, a player may have a zero marginal contribution, but, if it is needed to avoid a costly breakdown, it is not a dummy under our definition. This again reflects our philosophy that agreement involves all players in a given coalition. Our Additivity axiom has two parts. The first considers games with the same family of breakdown probabilities but different characteristic functions—and is the analog to Shapley Additivity. The second part of our Additivity axiom is new and considers games with the same characteristic function but different families of breakdown probabilities. This part is natural in our embellished setting in which a cooperative game comes along with these probabilities.

We offer two theorems on when the CCV overlaps or coincides with existing solution concepts. We prove that for the case of convex games, the CCV and the core have a nonempty intersection. We also prove that the CCV coincides with the Shapley value at every subset, for every characteristic function, if and only if the hitting probability of (i.e., the total probability of reaching) each subset of players is 1. This result captures the intuition (formalized by Shapley additivity) that the Shapley value effectively assumes that all component games of a given game are played in parallel.

The "catch" is that, in order to make all hitting probabilities equal 1, the family of breakdown probabilities must be signed, i.e., must allow for some negative weights. One such family is the system of upper-interval Möbius weights on the partition lattice of a subset (Stanley 53, 2012, Section 3.10). These weights can be understood as purely calculational—or they can be simulated by ordinary probabilities, as we explain later. We also explore two other salient families of signed breakdown weights, namely, the lower-interval Möbius weights and what we call the One-Big-Block family.

We have motivated and described our model as operating from the top of the probability graph down. Players ask what-if questions about breakdowns of coalitions and the analysis of these questions yields a division of the overall value. We can also operationalize this process by starting from the bottom of the graph. Now, the players start as singletons and, as they come together to form subsets, the associated gains are split equally among the relevant players. The probabilities of each of these upwards runs (to use later terminology) are derived from the original breakdown probabilities that operate downwards. It is easier to work with the top-down interpretation, but there is always an equivalent bottom-up process.

The CCV augments the structure of a cooperative game to include breakdown probabilities. This has the effect of endogenizing the partition formation, as in the above literature, but, in our case, the partitions

are in the "background" and serve to determine, along with the breakdown probabilities, the division of the overall value. Put differently, our partitions are counterfactual objects that come into play in setting the disagreement points for each coalition in the game. As for the breakdown probabilities, these may be exogenous or endogenous. We examine both cases—for example, exogenous in defining the CCV polytope and in the axiomatization, and endogenous in the loss-harmonic case and several examples.

In most of the paper, we allow the breakdown probabilities to be arbitrary. The CCV is a polytope of payoff vectors to the players, as we vary these probabilities. We see this indeterminacy as a natural expression of the notion that factors external to the characteristic function may affect a player's ability to influence these probabilities and hence the value they receive. Indeed, influence over breakdown probabilities seems a natural part of what can be meant by a player's bargaining ability. Despite this indeterminacy, we will see that in several games of interest, the CCV solution is highly circumscribed. The characteristic function may also influence the breakdown probabilities directly, which can be thought of as the endogenous vs. exogenous case. We study a salient family of endogenous probabilities where the chance that a subset breaks down to a particular partition is inversely proportional to the loss incurred.

The organization of the rest of the paper is as follows. Section 2 analyzes a series of motivating examples. Section 3 presents our formal framework. We define the CCV as a recursion formula on a directed acyclic graph and characterize the CCV in terms of the hitting probabilities of subsets. Section 4 compares the CCV with existing cooperative solution concepts. Section 5 is our axiomatization of the CCV. Section 6 proves that the CCV polytope has a nonempty intersection with the core in the case of convex games. Section 7 examines the CCV for several economic supply-and-demand scenarios modeled as cooperative games (Shapley and Shubik 50, 1969; Postlewaite and Rosenthal 39, 1974; Aumann 2, 2008) and for a voting game (Shapley and Shubik 49, 1954). In Section 8, we introduce signed breakdown weights, establish a formal relationship to the Shapley value, and revisit some earlier examples. Section 9 extends our framework to include cooperative games with externalities. In Section 10, we study other extensions. The first is to NTU games, where the CCV provides a route to endogenizing the disagreement points in the n-player versions of Nash (34, 1950) and Kalai and Smorodinsky (16, 1975). This is followed by two further axiomatizations of the CCV, one based on consistency (Hart and Mas-Colell 14, 1989) and the other on monotonicity (Young 58, 1985). We also provide a non-cooperative route to the CCV and re-derive the CCV as a least-squares minimization problem, paralleling the least-squares representation of the Shapley value known in both game theory (Charnes et al. 7, 1988) and machine learning (Lundberg and Lee 22, 2017). We discuss some conceptual aspects in Section 11. All proofs are in appendices, excepting a small number which are short and illuminating and are kept in the main text.

2 Motivating Examples

We present four examples to illustrate both the mechanics of calculating the CCV solution of a game and the motivation for the concept. We return in Section 7 to additional illustrations of our solution concept in a series of economic examples and a voting example.

Example 1. *Consider a three-player cooperative game with players a, b, c and characteristic function:*

$$v(\{a,b,c\}) = 1,\ v(\{a,b\}) = v(\{a,c\}) = \frac{3}{5},\ v(\{b,c\}) = \frac{4}{5}, \tag{1}$$

and $v(\{i\}) = 0$ for each player $i = a, b, c$. Absent a three-way agreement, there are four potential outcomes of a breakdown:

$$\{\{a\},\{b,c\}\},\ \{\{b\},\{a,c\}\},\ \{\{c\},\{a,b\}\},\ \{\{a\},\{b\},\{c\}\}, \tag{2}$$

where, in the first case, a two-way agreement forms between players b and c, and player a is a singleton, the second and third cases are analogous, and, in the fourth case, no agreement is reached among the players.

As an illustration, assume that the respective breakdown probabilities are $5/8, 1/6, 1/6$, and $1/24$. We give the highest probability to the first case, on the grounds that the $\{b,c\}$ coalition creates the highest value among pairs and might therefore be considered the most likely to form in the event of a breakdown. We give the lowest probability to the fourth case, on the grounds that it yields the lowest breakdown value. This is meant as a natural choice, but the CCV does not insist on this ranking.

The expected loss on breakdown is:

$$1 - \big[\frac{5}{8}\cdot(\frac{4}{5}+0) + \frac{1}{6}\cdot(\frac{3}{5}+0) + \frac{1}{6}\cdot(\frac{3}{5}+0) + \frac{1}{24}\cdot 0\big] = \frac{3}{10}. \tag{3}$$

As described in the Introduction, our logic is that since each of the players is needed to reach full agreement, the $3/10$ expected value is treated as it would be in a unanimity game. We therefore divide it equally among players a, b, and c.

Next, we treat each of the two-player subsets $\{b,c\},\{a,c\},\{a,b\}$ in exactly the same fashion. In the absence of a two-way agreement, each subset would break down with probability 1 to singletons, which do not create value. We therefore divide the value of each two-player coalition equally. The final expected payoffs are:

$$\phi_a = \frac{1}{10} + \frac{5}{8}\cdot 0 + \frac{1}{6}\cdot\frac{1}{2}\cdot\frac{3}{5} + \frac{1}{6}\cdot\frac{1}{2}\cdot\frac{3}{5} + \frac{1}{24}\cdot 0 = \frac{1}{5}, \tag{4}$$

$$\phi_b = \frac{1}{10} + \frac{5}{8}\cdot\frac{1}{2}\cdot\frac{4}{5} + \frac{1}{6}\cdot 0 + \frac{1}{6}\cdot\frac{1}{2}\cdot\frac{3}{5} + \frac{1}{24}\cdot 0 = \frac{2}{5}, \tag{5}$$

$$\phi_c = \frac{1}{10} + \frac{5}{8}\cdot\frac{1}{2}\cdot\frac{4}{5} + \frac{1}{6}\cdot\frac{1}{2}\cdot\frac{3}{5} + \frac{1}{6}\cdot 0 + \frac{1}{24}\cdot 0 = \frac{2}{5}. \tag{6}$$

Example 2. *We consider a three-player game which, at first blush, might be thought to yield the same answer as the Shapley value. It does not—and the reason helps illuminate the different philosophies underneath the two concepts. Assume $v(\{i\}) = 0$ for each i and let the breakdown probabilities (in the same order as in Example 1) be $1/3, 1/3, 1/3$, and 0—that is, uniform on the two-player breakdown coalitions. Write $l_i = v(N) - v(N\backslash\{i\})$. Direct calculation yields, for each player i:*

$$\phi_i^{\mathrm{CCV}} = \frac{1}{9}(l_a + l_b + l_c) + \frac{1}{6}\big[v(\{i,j\}) + v(\{i,k\})\big], \tag{7}$$

$$\phi_i^{\mathrm{Sh}} = \frac{1}{3}\cdot l_i + \frac{1}{6}\big[v(\{i,j\}) + v(\{i,k\})\big], \tag{8}$$

for the CCV and Shapley value, respectively. The term $(l_a + l_b + l_c)/9$ in the CCV comes from the fact that we first average ("collectivize") over the players the loss under breakdown and then allocate this quantity equally. The term $l_i/3$ in the Shapley value comes from the fact that we first allocate ("individualize") each loss to the respective player and then average this quantity over orderings equally. Differently put, the primitive for the CCV is what can be sustained only if a coalition stays together, while the primitive for the Shapley value is each separate player's marginal contribution to the whole.

Example 3. *Consider a special case of Example 2, a unanimity game with $v(\{a,b,c\}) = v(\{a,b\}) = 1$ and all other subsets are worth 0. The expected loss on breakdown is:*

$$1 - \big[\frac{1}{3}\cdot 0 + \frac{1}{3}\cdot 0 + \frac{1}{3}\cdot(1+0) + 0\cdot 0\big] = \frac{2}{3}. \tag{9}$$

The CCV divides the $2/3$ equally among the three players, so that each receives $2/9$. If the two-way coalition $\{a,b\}$ forms, which happens with probability $1/3$, players a and b share value of 1 equally. The overall expected payoffs are:

$$\phi_a = \phi_b = \frac{7}{18},\ \phi_c = \frac{2}{9}. \tag{10}$$

The CCV solution to this game coincides with the Solidarity value (Sprumont 52, 1990; Nowak and Radzik 36, 1994), although the two rationales are interestingly different. Nowak and Radzik argue that, while players a and b do not need player c, they should nevertheless subsidize this weaker player as an "act of solidarity". We argue that player c is, in fact, needed. There is a $2/3$ chance that breakdown will lead to 0, and therefore a loss of $2/3$ if any of the three players does not go along with a three-way agreement. In this regard, the positions of players a, b, and c are fully symmetric. They each have veto power over the three-way agreement. Player c is in a strong position in this game. Even though player c is a dummy in the sense of Shapley (45, 1953), it is not a dummy in our axiomatization of the CCV to come.

For a case where player c is truly superfluous, change the breakdown probabilities to put probability 1 on the partition $\{\{c\},\{a,b\}\}$. The idea is that players a and b can still create value of 1 without c. The CCV then becomes $(1/2, 1/2, 0)$. We will see that player c is now a dummy under our definition. Varying the breakdown probabilities, the range of payoffs to player c is 0 to $1/3$.[1]

Example 4. *We next consider a "deterministic" game in which only point (Dirac) breakdown probabilities occur. If the grand coalition $\{a,b,c\}$ breaks down, it does so to the partition $\{\{a\},\{b,c\}\}$ with probability 1. The next breakdown, of $\{b,c\}$ to the partition $\{\{b\},\{c\}\}$, is always deterministic. We leave the characteristic*

[1] The best case for player c arises when breakdown leads with probability 1 to any combination of $\{\{a\},\{b,c\}\}$, $\{\{b\},\{a,c\}\}$ or $\{\{a\},\{b\},\{c\}\}$. The loss is then 1, which is shared equally by the three players, so that c gets $1/3$.

function in its general form. Direct calculation yields:

$$\phi_a = \frac{1}{3}\big[v(\{a,b,c\}) - (v(\{b,c\}) + v(\{a\}))\big] + v(\{a\}), \tag{11}$$

$$\phi_b = \frac{1}{3}\big[v(\{a,b,c\}) - (v(\{b,c\}) + v(\{a\}))\big] + \frac{1}{2}\big[v(\{b,c\}) - (v(\{b\}) + v(\{c\}))\big] + v(\{b\}), \tag{12}$$

$$\phi_c = \frac{1}{3}\big[v(\{a,b,c\}) - (v(\{b,c\}) + v(\{a\}))\big] + \frac{1}{2}\big[v(\{b,c\}) - (v(\{b\}) + v(\{c\}))\big] + v(\{c\}). \tag{13}$$

These expressions suggest that a deterministic game can be built up from more primitive games—what we will call basic deterministic games—with different characteristic functions. In this example, there are five relevant such basic games: (i) a game that gives the grand coalition value $v(\{a,b,c\}) - (v(\{b,c\}) + v(\{a\}))$ and all other subsets 0; (ii) a game that gives the coalition $\{b,c\}$ value $v(\{b,c\}) - (v(\{b\}) + v(\{c\}))$, the same value to the grand coalition and 0 everywhere else; and (iii) three games for $i \in \{a,b,c\}$, each of which gives the singleton $\{i\}$ the value $v(\{i\})$, the same value to all sets containing i, and 0 everywhere else.

When we come to our axiomatization of the CCV in Section 5, we will break a game down into basic deterministic games. Each of these games is immediately solved via the Efficiency, Symmetry, and Dummy axioms. This step is broadly parallel to the Shapley (45, 1953) decomposition and solution of component games in his framework. (The comparison has limits—we have already mentioned how the notions of a dummy player are different.) We then add basic deterministic games to get back a deterministic game. This step is parallel to Shapley Additivity. Finally, we take an expectation over deterministic games, under the breakdown probabilities, to arrive at the solution of an original subset-refinement game. This step does not have an analog in the Shapley approach, because it does not have breakdown probabilities and effectively supposes all component games are played in parallel. We discuss parallelism in detail later.

3 The Subset-Refinement Game

We now present our formal framework and define the CCV formally. Let N be a finite set. For every subset $S \subseteq N$ with $|S| \geq 2$, define the set of all nontrivial partitions of S:

$$\Pi(S) = \{\pi : \pi \text{ is a partition of } S \text{ with } |\pi| \geq 2\}. \tag{14}$$

With this, we define two families of vertices of a directed graph:

$$V_1 = \{S \subseteq N : S \neq \emptyset\}, \tag{15}$$

$$V_2 = \{(S,\pi) : S \subseteq N,\ |S| \geq 2,\ \pi \in \Pi(S)\}. \tag{16}$$

Here, the first family of vertices comprises "subset vertices" and the second family comprises "partition vertices". Next are the two families of edges:

$$E_1 = \{\big(S,(S,\pi)\big) : S \in V_1,\ |S| \geq 2,\ \pi \in \Pi(S)\}, \tag{17}$$

$$E_2 = \{\big((S,\pi),T\big) : (S,\pi) \in V_2,\ T \in \pi\}. \tag{18}$$

The first family of edges goes from subset vertices to partition vertices, and the second family goes from partition vertices to subset vertices.

Definition 1. *A **subset-refinement graph** is a directed graph $G = (V,E)$ with vertices $V = V_1 \cup V_2$ and edges $E = E_1 \cup E_2$.*

It is easily checked that this directed graph is acyclic. We next specify a family of probability measures $p = (p_S)$, for $S \subseteq N$ with $|S| \geq 2$, that gives, for each nonsingleton subset S, the probability that breakdown leads to a particular partition of S. Thus, for all such subsets S:

$$\sum_{\pi \in \Pi(S)} p_S(\pi) = 1. \tag{19}$$

Note that p_S depends only on S. In particular, it does not depend on the path in the graph by which S was reached.[2]

[2]Corollary B.1 in Appendix B shows that this assumption can be dropped without changing the solution set. Note that the probability measure p_S lives on $b_{|S|} - 1$ points, where the Bell number $b_{|S|}$ is the number of partitions of a set of size $|S|$. We subtract 1 because we do not allow the trivial partition $\{S\}$.

Definition 2. *A **subset-refinement game** is a structure $\mathcal{G} = (N, v, G, p)$ where (N, v) is a TU cooperative game.*

Given a partition π and subset S, let π_S denote the member of π that contains S, where we set $\pi_S = \emptyset$ if there is no such member. For i in S, we write π_i for $\pi_{\{i\}}$ and note that $\pi_i \neq \emptyset$.

Definition 3. *Fix a subset-refinement game $\mathcal{G} = (N, v, G, p)$. The **co-creation value (CCV)** of $\mathcal{G}$, which we denote by $\phi^{\mathcal{G}}$, associates to each nonsingleton subset S and player i, where $i \in S$, the value $\phi_i^{\mathcal{G}}(S)$ given by the one-step recursion:*

$$\phi_i^{\mathcal{G}}(S) = \sum_{\pi \in \Pi(S)} p_S(\pi)\big[\frac{v(S) - \sum_{T \in \pi} v(T)}{|S|} + \phi_i^{\mathcal{G}}(\pi_i)\big], \tag{20}$$

with initial condition $\phi_i^{\mathcal{G}}(\{i\}) = v(\{i\})$.

Induction on the size of S shows that each $\phi_i^{\mathcal{G}}$ is uniquely defined by this formula. Since all of the losses are distributed, nothing leaves the system. For this reason, the CCV is efficient. In the formal statement of this result and onward, we write ϕ for $\phi^{\mathcal{G}}$ when the game in question is clear.

Proposition 1. *Efficiency holds at each subset S: $\sum_{i \in S} \phi_i(S) = v(S)$.*

Next, for a nonsingleton subset T of N and a partition π in $\Pi(T)$, set:

$$l_\pi(T) = v(T) - \sum_{U \in \pi} v(U), \tag{21}$$

and, for any nonsingleton T, write:

$$m(T) = \sum_{\pi \in \Pi(T)} p_T(\pi) l_\pi(T), \tag{22}$$

with initial conditions $m(\{i\}) = v(\{i\})$ for all i. The quantity $l_\pi(T)$ is the (local) loss when partition π of T forms. The quantity $m(T)$ is the expectation of this loss, taken over all partitions π of T under the probabilities p_T. With these definitions, we can rewrite Equation 20 as:

$$\phi_i(S) = \frac{m(S)}{|S|} + \sum_{\pi \in \Pi(S)} p_S(\pi)\, \phi_i(\pi_i). \tag{23}$$

Example 5. *We illustrate our concept of a subset-refinement game and the recursion of Equation 23 for the CCV for the three-player case in Figure 1. Observe the alternating layers of subsets and partitions—and that the root and terminal nodes belong to subset layers. The numbers in red depict the breakdown probabilities starting from each nonsingleton subset.*

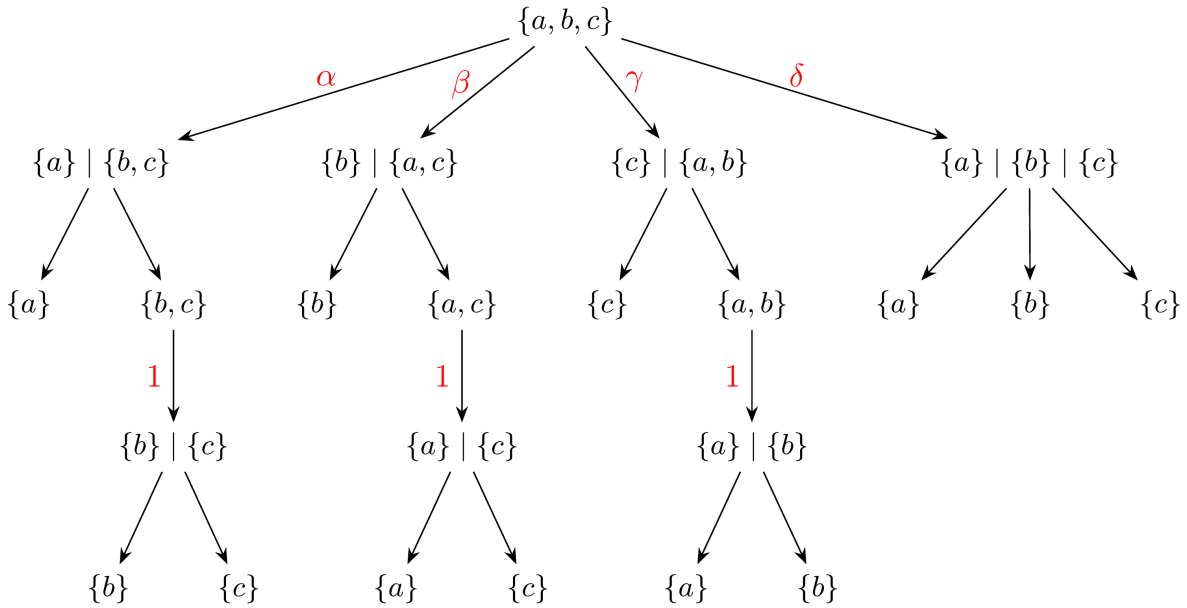

Figure 1: A Subset-Refinement Game

Applying Equation 23 and eliminating the parameter δ (using $\alpha+\beta+\gamma+\delta=1$), we find the following formula for what the CCV gives player a:

$$\phi_a(\{a,b,c\}) = \frac{m(\{a,b,c\})}{3} + \beta\frac{m(\{a,c\})}{2} + \gamma\frac{m(\{a,b\})}{2} + v(\{a\}). \tag{24}$$

The formulas for players b and c are analogous.

Next, for $T \subsetneq S$, let $\lambda_S(T)$ denote the total probability that the set T is reached, starting from S. We set $\lambda_T(T)=1$ and, for $T \not\subseteq S$, we set $\lambda_S(T)=0$. In the usual terminology, $\lambda_S(T)$ is the hitting probability of T, starting from S. For $T \subsetneq S$, it obeys the one-step recursion:

$$\lambda_S(T) = \sum_{\pi\in\Pi(S)} p_S(\pi)\,\lambda_{\pi_T}(T). \tag{25}$$

Note that $\lambda_S(\{i\})=1$ for all i in S. (This follows easily by induction on $|S|$, since every partition of S contains a unique cell containing i.)

Example 5 contd. *We can illustrate the hitting-probability recursion by writing:*

$$\begin{aligned}
\lambda_{\{a,b,c\}}(\{a\}) &= \alpha\,\lambda_{\{a\}}(\{a\}) + \beta\,\lambda_{\{a,c\}}(\{a\}) + \gamma\,\lambda_{\{a,b\}}(\{a\}) + \delta\lambda_{\{a\}}(\{a\}) && (26)\\
&= \alpha\cdot 1 + \beta\cdot 1 + \gamma\cdot 1 + \delta\cdot 1 = 1, && (27)\\
\lambda_{\{a,b,c\}}(\{a,b\}) &= \gamma\,\lambda_{\{a,b\}}(\{a,b\}) && (28)\\
&= \gamma\cdot 1 = \gamma, && (29)
\end{aligned}$$

with analogous calculations for players b and c.

We now provide an alternative formula for the CCV. For a player i in S, define:

$$\phi_i^*(S) = \sum_{i\in T\subseteq S} \lambda_S(T)\frac{m(T)}{|T|}. \tag{30}$$

This alternative formula follows the recursion in Equation 23 to find all the instances where $m(T)/|T|$ appears.

Proposition 2. *For each subset S and player i in S, we have $\phi_i^*(S)=\phi_i(S)$.*

Recall that the Shapley value can be written in the Harsanyi dividend (12, 1959) form:

$$\phi_i^{\mathrm{Sh}}(S) = \sum_{i\in T\subseteq S} \frac{D(T)}{|T|}, \tag{31}$$

where the dividends $D(T)$ are the Möbius inversion of the characteristic function and are defined by the recursion:

$$D(T) = v(T) - \sum_{U\subsetneq T} D(U). \tag{32}$$

Equations 30 and 31 bear an obvious similarity. Both involve an equal split of a notion of gain among the players responsible for it. However, the CCV dividend is $\lambda_S(T)\,m(T)$, which, in general, is different from the Harsanyi dividend $D(T)$. (We saw instances of different answers in the previous section.) Once again, the issue is the parallelism of the Shapley value, which effectively assumes all games are played simultaneously. We show in Section 8 that if all $\lambda_S(T)=1$, then $m(T)=D(T)$, from which it follows that the two dividend concepts are identical and the CCV coincides with the Harsanyi formula for the Shapley value.

We next note that, under superadditivity, the CCV obeys individual rationality.

Proposition 3. *Suppose the characteristic function v is superadditive. Then, for every nonempty subset S and every player i in S:*

$$\phi_i(S) \geq v(\{i\}). \tag{33}$$

Finally in this section, we verify that the CCV is affine invariant and therefore can be zero-normalized without loss of generality, as is standard for cooperative solution concepts.

Proposition 4. *Fix subset-refinement games $\mathcal{G}=(N,v,G,p)$ and $\mathcal{G}'=(N,v',G,p)$ where, for each S and $i\in S$:*

$$v'(S) = \mu\cdot v(S) + \sum_{i\in S}\nu_i, \tag{34}$$

for parameters $\mu \geq 0$ and ν_i with $i \in N$. Then $\phi'_i(S) = \mu\phi_i(S) + \nu_i$.

4 Relationship to Existing Solution Concepts

In this section, we describe the geometry of the CCV solution concept and, in a numerical example, we compare it with various existing cooperative solution concepts. To establish the geometry of the CCV, fix a player set N, a characteristic function v, and a subset-refinement graph G, and write the set of all families of breakdown probability measures p as:

$$\mathcal{B} = \prod_{\{S \subseteq N\,:\,|S| \geq 2\}} \Delta(\Pi(S)), \tag{35}$$

where $\Delta(\Pi(S))$ is the set of all probability measures on the set $\Pi(S)$ of nontrivial partitions of S. Next, for the given characteristic function v, let $w(p)$ be the vector of expected payoffs to the players under the CCV, where we make explicit the dependence on p. Write:

$$\mathcal{W} = \{w(p) : p \in \mathcal{B}\} \subseteq \mathbb{R}^{|N|}. \tag{36}$$

The set of all families of deterministic breakdown probabilities is:

$$\mathcal{D} = \{p \in \mathcal{B} : p_S \text{ is a Dirac measure for each } S\}, \tag{37}$$

and we write:

$$\mathcal{V} = \{w(p) : p \in \mathcal{D}\}. \tag{38}$$

Theorem 1. *Fix a cooperative game (N, v) and the associated graph G. Then $\mathcal{W}$ is the convex hull of $\mathcal{V}$, that is: $\mathcal{W} = \text{conv}(\mathcal{V})$.*

In words, for a fixed cooperative game (N, v) and the associated graph G, the set of CCV expected-payoff vectors as we vary the family of breakdown probability measures p is a polytope in $\mathbb{R}^{|N|}$. Corollary B.1 establishes a stronger result. Suppose we enhance the definition of a subset-refinement game by allowing the breakdown probability measures $p_S(\pi)$ to vary according to the occurrence of S in the graph—and then define the enhanced CCV by allowing this dependence in Equation 19 onward. The set of attainable CCV payoff vectors remains unchanged. Allowing breakdown probabilities to be contextual offers an apparent but not an actual increase in generality.

We now consider a numerical example to illustrate the solution polytope and compare it with several existing solution concepts.

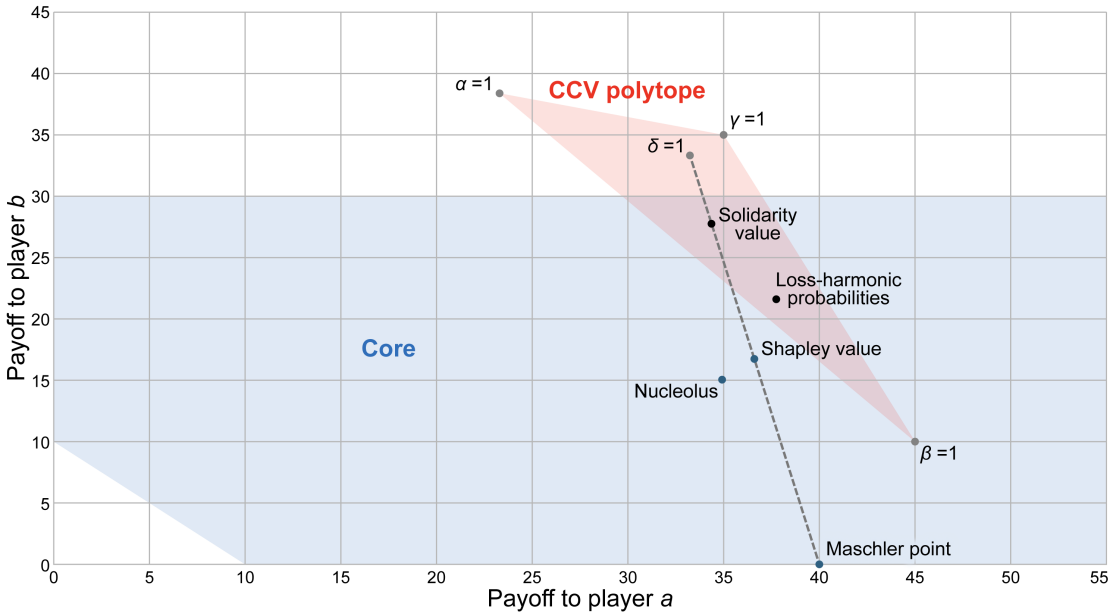

Figure 2: Solution Concepts in a Convex Game

Example 6. *Consider the three-player game with characteristic function:*

$$v(\{a,b,c\}) = 100,\ v(\{a,b\}) = 10,\ v(\{b,c\}) = 30,\ v(\{a,c\}) = 70, \tag{39}$$

and $v(\{i\}) = 0$ *for each* $i = a, b, c$. *This is a convex game, that is:* $v(S \cup T) \geq v(S) + v(T) - v(S \cap T)$. *Using the breakdown probability measure* $(\alpha, \beta, \gamma, \delta)$ *from Example 5, we can calculate:*

$$m(\{a,b,c\}) = 70\alpha + 30\beta + 90\gamma + 100\delta, \tag{40}$$

$$m(\{a,b\}) = 10,\ m(\{b,c\}) = 30,\ m(\{a,c\}) = 70, \tag{41}$$

and $m(\{i\}) = 0$ *for each player* i. *The hitting probabilities are:*

$$\lambda_N(\{a,b,c\}) = 1,\ \lambda_N(\{a,b\}) = \gamma,\ \lambda_N(\{b,c\}) = \alpha,\ \lambda_N(\{a,c\}) = \beta, \tag{42}$$

and $\lambda_N(\{i\}) = 1$ *for each* i. *Using Proposition 2, we can write down the CCV:*

$$\phi_a(N) = \frac{1}{3}(70\alpha + 30\beta + 90\gamma + 100\delta) + \frac{1}{2}(70\beta + 10\gamma), \tag{43}$$

$$\phi_b(N) = \frac{1}{3}(70\alpha + 30\beta + 90\gamma + 100\delta) + \frac{1}{2}(30\alpha + 10\gamma), \tag{44}$$

$$\phi_c(N) = \frac{1}{3}(70\alpha + 30\beta + 90\gamma + 100\delta) + \frac{1}{2}(30\alpha + 70\beta). \tag{45}$$

Figure 2 shows the projection of the CCV polytope for this game onto the player-a and player-b axes.

The CCV solution at $\delta = 1$ divides the total value over the standalone values equally among the players and adds this quantity to each player's standalone value. This solution was first studied as the Equal Surplus Division rule by Driessen and Funaki (9, 1991) and is also sometimes called the TU egalitarian solution. In our example:

$$\phi_a^{\mathrm{Eq}} = \phi_b^{\mathrm{Eq}} = \phi_c^{\mathrm{Eq}} = 33\tfrac{1}{3}. \tag{46}$$

The CCV solution at $\alpha = \beta = \gamma = 1/3$ coincides with the Solidarity value (Nowak and Radzik 36, 1994) and gives the players:

$$\phi_a^{\mathrm{So}} = 34\tfrac{4}{9},\ \phi_b^{\mathrm{So}} = 27\tfrac{7}{9},\ \phi_c^{\mathrm{So}} = 37\tfrac{7}{9}, \tag{47}$$

which is also the center of gravity of the polytope. The Solidarity value was pioneered by Sprumont (52, 1990) as an example of a Population Monotonic Allocation Scheme, rediscovered, named the Solidarity value, and axiomatized by Nowak and Radzik (36, 1994), and then proceduralized by Malawski (24, 2013).[3] In the Solidarity value, the individual marginal contribution at a coalition is replaced by the average marginal contribution of all members of that coalition; this coalition-average amount is then used in the value formula. The CCV coincides with the Solidarity value when: (i) we give probability $1/|S|$ to each partition of S where one player is left out (and 0 elsewhere), and (ii) all standalone values are 0. The CCV differs in requiring affine invariance.[4]

The Shapley value for the game of Example 6 is:

$$\phi_a^{\mathrm{Sh}} = 36\tfrac{2}{3},\ \phi_b^{\mathrm{Sh}} = 16\tfrac{2}{3},\ \phi_c^{\mathrm{Sh}} = 46\tfrac{2}{3}, \tag{48}$$

which lies outside the CCV polytope. However, the Shapley value can be obtained within the CCV framework if signed breakdown weights are admitted. In particular, setting $\alpha = \beta = \gamma = 1$ and $\delta = -2$ in Equations 43–45 yields the payoff vector of Equation 48. We return to signed realization of the Shapley value in Section 8. The nucleolus (Schmeidler 44, 1969) is another well-known cooperative solution concept. It is the point:

$$\phi_a^{\mathrm{Nu}} = 35,\ \phi_b^{\mathrm{Nu}} = 15,\ \phi_c^{\mathrm{Nu}} = 50, \tag{49}$$

and lies outside the CCV polytope in our example. Maschler (25, 1963) considered games where the value-

[3] Nowak and Radzik (36, 1994) show that the Solidarity value is uniquely defined by the Shapley Efficiency, Symmetry, and Additivity axioms, together with a Null-Player axiom. A player is null in their sense if the average marginal contribution of every subset to which it belongs is 0, in which case the requirement is that the player gets 0. Under the Malawski (24, 2013) procedure, players join in a random order, as in the Shapley procedure, but the joining player's marginal contribution is shared equally between the joining player and each of the members in the coalition being joined.

[4] Consider a two-player game with $v(\{a,b\}) = 2, v(\{a\}) = 1, v(\{b\}) = 0$. The Solidarity value is $(5/4, 3/4)$, thereby sharing with player b part of player a's standalone advantage. If we first zero-normalize the game, find the Solidarity value, and then add back the standalone values, the solution is $(3/2, 1/2)$. Now, standalone values are treated as prior entitlements, and only the gain is shared. The CCV is neutral to baselines (Proposition 4) and is $(3/2, 1/2)$ in both cases. The distinction arises because the Solidarity value shares marginal contributions, while the CCV shares gains relative to breakdown; see also Example 3.

creating coalitions are the grand coalition N, the $(|N|-1)$-player coalitions, and individual players, and his solution concept serves to equalize gains from $(|N|-1)$-player deviations. Our example fits his format and its Maschler value is:

$$\phi_a^{\text{Ma}} = 40,\ \phi_b^{\text{Ma}} = 0,\ \phi_c^{\text{Ma}} = 60. \tag{50}$$

It, too, lies outside the CCV polytope. In our three-player example, it can be obtained as a CCV with signed breakdown weights $\alpha = \beta = \gamma = 2$ and $\delta = -5$.

There is an important geometric feature in the three-player case. The Equal Surplus Division point, the Solidarity value, the Shapley value, and the Maschler value all lie on a single line (refer back to Figure 2). These are all symmetric solutions. In general, the symmetric solutions are a much smaller solution set inside the polytope. In the three-player case, the symmetric solutions are characterized by weights $\alpha = \beta = \gamma = x$ and $\delta = 1 - 3x$, so that the probabilities sum to 1. Inside the polytope, the range of symmetric solutions is $x \in [0, 1/3]$.[5]

The Equal Surplus Division point is the case $x = 0$, the Solidarity value is the case $x = 1/3$, the Shapley value is the case $x = 1$, and the Maschler value is the case $x = 2$. We also see that the Shapley value lies exactly halfway between the Equal Surplus Division point and the Maschler value.

The Shapley value is the last point on the line towards the Maschler value that ensures a dummy player is not pushed below its standalone value. Suppose the game is essential, that is: $v(N) > \sum_i v(\{i\})$. Then $\phi_i^{\text{Eq}} > v(\{i\})$ and since payoffs vary affinely with x, we also have:

$$\phi_i(x) = (1-x)\,\phi_i^{\text{Eq}} + x\,\phi_i^{\text{Sh}}. \tag{51}$$

A Shapley dummy player i receives $v(\{i\})$, which is the value $\phi_i(1)$. For $x > 1$, it follows that $\phi_i(x) < v(\{i\})$.

In summary, there is a tradeoff between the "individually-protective" solution represented by the Equal Surplus Division and the "coalition-protective" solution represented by the Maschler value. The Equal Surplus solution does the best job reducing the incentive for any individual player to deviate (since it maximizes the minimum gain over acting alone). However, this makes it attractive for the two players with the greatest $v(\{i,j\})$ to deviate. At the other extreme, the Maschler point creates the lowest possible incentive for any pair of players to deviate, but at the cost of having low incentives for individual players to participate, and thereby potentially violating individual rationality constraints. Inside the polytope, the CCV embraces this tradeoff up to $x = 1/3$ (since all probabilities must be non-negative). The Shapley value goes further along the tradeoff, while still respecting individual participation constraints.

Finally, we see that in Example 6, the CCV and the core have a nonempty intersection. (Also, since the game is convex, the Shapley value lies in the core; see Shapley 48, 1971.) This is an instance of a general result we prove in Section 6. We exhibit a specific point in the CCV polytope, where the breakdown probabilities are endogenously defined from the characteristic function, which we prove (Theorem 3) lies in the core whenever the game is convex. We believe these probabilities are of independent interest and will discuss them in some detail later.

5 Axiomatization

In this section, we provide an axiomatization of the CCV.

Definition 4. *Consider the family of subset-refinement games $\mathcal{G} = (N, v, G, p)$. A **solution concept** is a map $\mathcal{G} \mapsto \psi(\mathcal{G})$, where $\psi(\mathcal{G}) \in \mathbb{R}^{|N|}$ and $\psi_i(\mathcal{G})$ is interpreted as the payoff to player i.*

Axiom 1. ***(Efficiency)*** *The grand coalition N satisfies $\sum_{i \in N} \psi_i(\mathcal{G}) = v(N)$.*

Axiom 2. ***(Symmetry)*** *If, for all $S \subseteq N$ with $i, j \notin S$:*

$$\lambda_N(S \cup \{i\}) \times m(S \cup \{i\}) = \lambda_N(S \cup \{j\}) \times m(S \cup \{j\}), \tag{52}$$

then $\psi_i(\mathcal{G}) = \psi_j(\mathcal{G})$.

We illustrate the first two axioms.

Example 7. *Consider the three-player game where $v(\{a,b,c\}) = 1$ and all other coalitions are worth 0. The breakdown probabilities are $1/2$ for the partition $\{\{a\},\{b,c\}\}$, $1/3$ for the partition $\{\{b\},\{a,c\}\}$, and $1/6$ for the partition $\{\{c\},\{a,b\}\}$. The only subsets S that do not include a and b are $\emptyset$ and $\{c\}$. Both*

[5] Recall that Figure 2 is the projection of the CCV onto the a- and b-axes. This is why it looks like the CCV points in the form of Equation 51 extend past the Solidarity value to $x > 1/3$. They do not.

$m(\{a\}) = m(\{b\}) = 0$ *and* $m(\{c\} \cup \{a\}) = m(\{c\} \cup \{b\}) = 0$. *Therefore, Symmetry requires that players a and b receive the same payoff. It might seem that the positions of players a and b are asymmetric, since the player a is left out with a larger probability than is player b. But, each player is needed for a three-way agreement, and, absent an agreement, the value falls to* 0 *regardless of path. In our framework, players a and b are, in fact, in symmetric positions. The same argument applies to players b and c. By Efficiency, each player therefore receives* 1/3.

We next define the concept of a dummy player in our framework.

Definition 5. *Given a game* $\mathcal{G}$*, we say player* i *is a* ***dummy*** *if for every nonsingleton subset* $U \subseteq N$ *with* $i \in U$*, we have* $\lambda_N(U)\, m(U) = 0$.

Axiom 3. ***(Dummy)*** *If player* i *is a dummy, then* $\psi_i(\mathcal{G}) = v(\{i\})$.

This definition of a dummy player is different from a Shapley (45, 1953) dummy, who adds only its standalone value to any set: $v(S \cup \{i\}) = v(S) + v(\{i\})$. A Shapley dummy need not be a dummy in our sense as seen in Example 3. However, an important sufficient condition for this direction is that $\lambda_S(T) = 1$ for all nonempty $T \subseteq S \subseteq N$. This is a corollary to Theorem 4 to come. In the other direction, if player i is a dummy in our sense for all breakdown probabilities, then it is a Shapley dummy.

For our final axiom, fix games $\mathcal{G}^k = (N, v^k, G, p^k)$ and $\alpha^k \geq 0$, for $k = 1, 2, \ldots, K$. Note that we hold the player set N, and therefore the graph G, fixed. We define a new game $\mathcal{G} = (N, v, G, p)$ as the formal addition of these games in two cases. In the first case, $\alpha^k = 1$ for all k, and the characteristic functions v^k can differ, but the breakdown probability families are the same: $p^k_S(\pi) = p_S(\pi)$ for all k, π, and S. We define:

$$v(S) = \sum_{k=1}^{K} \alpha^k\, v^k(S). \tag{53}$$

In the second case, we require $\sum_k \alpha^k = 1$. We let the breakdown probability families p^k differ, but the characteristic functions are the same: $v^k(S) = v(S)$ for all k. Define:

$$\lambda_N(S) = \sum_{k=1}^{K} \alpha^k \lambda^k_N(S), \tag{54}$$

and:

$$p_S(\pi) = \frac{\sum_{k=1}^{K} \alpha^k \lambda^k_N(S)\, p^k_S(\pi)}{\lambda_N(S)}, \tag{55}$$

provided $\lambda_N(S) > 0$, where $p_S(\pi)$ is arbitrary (while summing to 1 over π in $\Pi(S)$ otherwise). In both cases, we write the formal addition as:

$$\mathcal{G} = \bigoplus_{k=1}^{K} \alpha^k\, \mathcal{G}^k. \tag{56}$$

Axiom 4. ***(Additivity)*** *Under each case, we have:*

$$\psi_i\Big(\bigoplus_{k=1}^{K} \alpha^k\, \mathcal{G}^k\Big) = \sum_{k=1}^{K} \alpha^k\, \psi_i(\mathcal{G}^k). \tag{57}$$

Theorem 2. *There is a unique solution concept* ψ *satisfying Axioms 1–4 given by* $\psi_i(\mathcal{G}) = \phi_i(N)$ *for every player* i.

The proof is in Appendix C. Here, we give the outline of the proof in the form of a sequence of lemmas.

Definition 6. *A run* r *specifies, for each nonsingleton subset* S *in the graph, a partition of* S*. Equivalently, a run is a family of breakdown probability measures where each measure is Dirac.*

Notice that, in game-theoretic language, a run is a complete pure strategy of Nature that specifies a partition even at subsets that are not reached on the run. A family of breakdown probability measures induces a probability measure on runs via multiplication in the obvious way. Given a run r and a subset S, let $\pi^r(S) \in \Pi(S)$ denote the partition of S selected on the run.

Definition 7. *The game* $\mathcal{G} = (N, v, G, r)$ *is the* ***deterministic game*** *associated with run* r *and will be denoted by* $\mathcal{G}^r$.

The original game $\mathcal{G}$ can be viewed as a mixture of the deterministic games $\mathcal{G}^r$, weighted by the probability with which each run r occurs. Next, given a run r and a nonempty subset U of N, let $l_r(U)$ be the local loss at U:

$$l^r(U) = v(U) - \sum_{T \in \pi^r(U)} v(T), \tag{58}$$

when $|U| \geq 2$, and set $l^r(\{i\}) = v(\{i\})$ for every i.

We define a basic deterministic game $\mathcal{G}^r_U = (N, v^r_U, G, r)$ with characteristic function v^r_U as follows. The subset U is assigned value $l^r(U)$. Consider a superset W of U such that the path from W to U receives probability 1 under r. This subset W is also assigned the value $l^r(U)$. All other subsets of N receive value 0. This defines a characteristic function v^r_U.

Definition 8. *Given a run r and a subset $U \subseteq N$, the game $\mathcal{G}^r_U = (N, v^r_U, G, r)$ is called the associated **basic deterministic game**.*

We first solve basic deterministic games.

Lemma 1. *Fix a basic deterministic game $\mathcal{G}^r_U = (N, v^r_U, G, r)$. For every player $k \in N$, the payoff $\psi_k(\mathcal{G}^r_U)$ is given as follows:*

(i) If $k \notin U$, then $\psi_k(\mathcal{G}^r_U) = 0$;

(ii) If $k \in U$ then $\psi_k(\mathcal{G}^r_U) = \lambda_N(U)\, \frac{l_r(U)}{|U|}$.

The next step is to reassemble a deterministic game from basic deterministic games. Given a run r and subset W, let $\mathcal{R}(r, W)$ denote the family of subsets that are reached on r starting from W.

Lemma 2. *A deterministic game $\mathcal{G}^r$ decomposes as the sum of its basic deterministic games:*

$$\mathcal{G}^r = \bigoplus_{\emptyset \neq U \subseteq N} \mathcal{G}^r_U. \tag{59}$$

Equivalently, for every subset $W \subseteq N$:

$$v(W) = \sum_{\emptyset \neq U \subseteq N} v^r_U(W). \tag{60}$$

We now derive the payoffs in a given deterministic game.

Lemma 3. *Fix a run r. For each player i:*

$$\psi_i(\mathcal{G}^r) = \sum_{i \in U \in \mathcal{R}(r,N)} \frac{l_r(U)}{|U|}. \tag{61}$$

The final step is to average over runs to turn the run-level payoffs into the CCV formula.

Lemma 4. *For each player i:*

$$\psi_i(\mathcal{G}) = \sum_{i \in U \subseteq N} \frac{\lambda_N(U)\, m(U)}{|U|}. \tag{62}$$

The proof of Theorem 2 is now complete, since Equation 62 is the formula for $\phi^*_i(N)$ which, by Proposition 2, is equal to $\phi_i(N)$.

As a by-product, we can use our concept of a run (Definition 6) employed in the proof of Theorem 2 to give a procedure that operationalizes the CCV. Recall that the Shapley value can be operationalized as a random-order procedure in which players enter one at a time, and each entering player receives its marginal contribution to the coalition of players already present. Formally, for each player i, the Shapley value can be written as:

$$\phi^{\mathrm{Sh}}_i = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!\,(|N| - |S| - 1)!}{|N|!} \big[v(S \cup \{i\}) - v(S)\big]. \tag{63}$$

In our case, fix a subset-refinement game $\mathcal{G} = (N, v, G, p)$. Read downwards, the graph G depicts the counterfactual breakdowns that determine the division of the total value $v(N)$. By starting at the bottom of the graph, we can build an equivalent process that is directly playable. For example, in the three-player case, an upward run starts from the singletons $\{a\}, \{b\}, \{c\}$. The upward run then reverses a possible run. Thus, if the downward run is:

$$\{a, b, c\} \to \{a\}, \{b, c\} \to \{a\}, \{b\}, \{c\}, \tag{64}$$

then the upward run is:

$$\{a\},\{b\},\{c\} \to \{a\},\{b,c\} \to \{a,b,c\}. \tag{65}$$

Each player begins with its standalone value. Then, when players b and c coalesce, they split the gain

$$v(\{b,c\}) - v(\{b\}) - v(\{c\}) \tag{66}$$

equally. Next, player a joins the already-formed pair $\{b,c\}$, but the gain:

$$v(\{a,b,c\}) - v(\{a\}) - v(\{b,c\}), \tag{67}$$

is split equally among all three players (not just to a). In this way, each upward run gives a well-defined total allocation to each player. The CCV is the expectation over these allocations, taken under the probabilities of the original (downward) runs. Notice that, in general, when there are more than three players, two or more coalescence events can happen simultaneously on an upward run. This causes no difficulty because the simultaneity happens on distinct paths.

As expected, our operationalization of the CCV differs from the standard operationalization of the Shapley value (reviewed above) in, once again, treating gains as a collective to be shared equally by members of a subset that forms rather than as an amount that is calculated on an individual-player basis and fully awarded to that player.

6 Relationship to the Core

In this section, we establish that the CCV and the core have a nonempty intersection when the game (N,v) is convex. We prove this by construction using a family of loss-harmonic breakdown probability measures, which we define for the case of superadditive games.

For each nonsingleton subset S and $i \in S$, let $l^i(S) = v(S) - [v(S\backslash\{i\}) + v(\{i\})]$ denote the loss created by the partition in which player i is left out. Superadditivity implies that $l^i(S) \geq 0$. We denote the harmonic mean of the losses by:

$$H(S) = \begin{cases} \dfrac{|S|}{\sum_{i\in S} 1/l^i(S)} & \text{if } l^i(S) > 0 \text{ for all } i, \\ 0 & \text{otherwise.} \end{cases} \tag{68}$$

Definition 9. *A family of breakdown probability measures p is **loss-harmonic (LH)** if for every subset S with $|S| \geq 3$:*

$$p_S(\pi) = \begin{cases} \dfrac{H(S)}{|S|} \cdot \dfrac{1}{l^i(S)} & \textit{if } \pi = \{\{i\}, S\backslash\{i\}\}, \\ 0 & \textit{otherwise,} \end{cases} \tag{69}$$

whenever $l^i(S) > 0$ for all $i \in S$. If not, let $\mathcal{N}(S) = \{i \in S : l^i(S) = 0\}$ and set:

$$p_S(\pi) = \begin{cases} \dfrac{1}{|\mathcal{N}(S)|} & \textit{if } i \in \mathcal{N}(S) \textit{ and } \pi = \{\{i\}, S\backslash\{i\}\}, \\ 0 & \textit{otherwise.} \end{cases} \tag{70}$$

We omit two-player subsets $S = \{i,j\}$ from this definition, since i and j effectively exit simultaneously to arrive at the two singletons. It will be helpful later to adopt the convention that, with probability $1/2$, player i exits before j, and, with probability $1/2$, player j exits before i.

An attractive feature of LH probabilities is that a Shapley dummy is also a dummy player in our sense.[6] A second attractive feature is that they identify a solution in the core for convex games.[7]

Theorem 3. *Let $\mathcal{G} = (N,v,G,p)$ be a subset-refinement game where (N,v) is convex and p is loss-harmonic. The (unique) CCV lies in the core of the cooperative game (N,v).*

The intuition for this result is that LH probabilities make the CCV's division of gains coincide, in expectation, with a marginal-contribution procedure. To see this, observe that LH probabilities put weight only on one-player exits. Taking each sequence of exits in reverse order yields a player ordering. At a

[6] In the notation of this section, player i is a Shapley dummy if $l^i(S) = 0$ whenever i lies in S. Then, Equation 70 determines the breakdown probabilities $p_S(\pi)$. From this, $m(S) = 0$ and so $\lambda_N(S)\, m(S) = 0$ for all S where i lies in S. This says that player i is a dummy in the sense of our Definition 5.

[7] In the absence of convexity, the CCV and the core need not overlap. Consider the three-player balanced (Bondareva 6, 1963; Shapley 46, 1967) non-convex game where $v(\{a,b,c\}) = v(\{a,b\}) = v(\{a,c\}) = 1$, and all other subsets are worth 0. The core is the single point $(1,0,0)$, which is not attainable since the maximal payoff to a in the CCV is $1/2$.

subset S, if player i is the next player left out, it is last in the corresponding ordering of S, and the marginal-contribution procedure gives i:

$$v(S) - v(S\backslash\{i\}) = l^i(S) + v(\{i\}). \tag{71}$$

For notational convenience, write:

$$q_i(S) = p_S(\{\{i\}, S\backslash\{i\}\}), \tag{72}$$

when $|S| \geq 3$ and, when $S = \{i, j\}$, set $q_i(S) = q_j(S)$ in accordance with our earlier convention. Under LH probabilities, player i's expected local gain above $v(\{i\})$, namely, $q_i(S)\, l^i(S)$, is the same for every i in S. Since these gains sum to $m(S)$, each is equal to $m(S)/|S|$, exactly as under the CCV. If some loss is 0, LH probabilities select only zero-loss exits and the same conclusion holds with $m(S) = 0$. The LH CCV is therefore a convex combination of marginal vectors. For a convex game, every marginal vector lies in the core (Shapley 48, 1971; Weber 57, 1988). Since the core is convex, the LH CCV also lies in the core.

We end this section by observing that, apart from their use in the proof of Theorem 3, LH probabilities are natural in their own right. They say that more costly breakdowns are systematically less likely to arise. Examples 9–11 in the next section will make assumptions on breakdown probabilities that are consistent with LH probabilities—though we leave some leeway in their exact values.

7 Applications

We examine how the CCV behaves in a series of well-studied economic scenarios and one voting scenario all modeled as cooperative games. We compare the results of the CCV with the core and the Shapley value. In these examples, we restrict attention to breakdown probabilities that are non-negative, so that all solutions lie in the CCV polytope.

Example 8. *(**The Glove Game**) This game is a special case of the market games of Shapley and Shubik (50, 1969) and is called the "glove game" in Aumann (2, 2008). Consider a market with B buyers and S sellers. Any pair made up of one buyer and one seller creates value of 1. Assume, without loss of generality, that $B \geq S$. We calculate the lower and upper bounds on the buyer and seller totals under the CCV.*

For the buyers in total, the worst case is when all the sellers remain and, once sufficient buyers have left to achieve balance in the market, the remaining buyers leave one at a time. Initially, there is an excess of buyers and thus no loss upon their exit. Thereafter, at the step from k buyers and S sellers to $k-1$ buyers and S sellers, for $k = S, \ldots, 2, 1$, the loss is 1 and the buyers' total share is $k/(S+k)$. Summing up yields:

$$\underline{\Phi}_B = 0 + \sum_{k=1}^{S} \frac{k}{S+k} = \sum_{k=1}^{S} (1 - \frac{S}{S+k}) = S - S\sum_{k=1}^{S} \frac{1}{S+k} \tag{73}$$

$$= S\big[1 - (H_{2S} - H_S)\big] = S(1 - H_{2S} + H_S), \tag{74}$$

where we write $\underline{\Phi}_B$ for the minimal sum of the payoffs to the buyers and H_s is the Sth harmonic number.

For total buyer value, the best case is when all buyers remain and the sellers leave one at a time. At the step from B buyers and k sellers to B buyers and $k-1$ sellers, for $k = S, \ldots, 2, 1$, the loss is 1 and the buyers' total share is $B/(B+k)$. Summing up yields:

$$\overline{\Phi}_B = \sum_{k=1}^{S} \frac{B}{B+k} = B(H_{B+S} - H_B), \tag{75}$$

where $\overline{\Phi}_B$ is the maximal sum of the payoffs to the buyers. From these calculations, we immediately get:

$$\underline{\Phi}_S = S - B(H_{B+S} - H_B), \tag{76}$$

$$\overline{\Phi}_S = S - S(1 - H_{2S} + H_S) = S(H_{2S} - H_S), \tag{77}$$

where $\underline{\Phi}_S$ and $\overline{\Phi}_S$ are the minimal and maximal sums of the payoffs to the sellers.

It is instructive to see the asymptotic behavior of the per-player payoffs. Use $H_n = \ln n + \gamma + O(1/n)$, where γ is the Euler-Mascheroni constant, and let $B, S \to \infty$ with $S/B \to \rho \in (0, 1]$. From Equations 73–77, the CCV intervals of per-buyer and per-seller payoffs are, respectively:

$$\big[\rho(1 - \ln 2),\, \ln(1+\rho)\big], \tag{78}$$

$$\big[1 - \frac{1}{\rho}\ln(1+\rho),\, \ln 2\big]. \tag{79}$$

As $\rho \to 0$, so that the proportional excess of buyers becomes very large, the per-buyer payoff goes to 0, as expected. More interesting is that the limiting interval for the per-seller payoff is $[0, \ln 2]$. We might have expected that the maximum per-seller payoff would go to 1, but that is supply-and-demand logic (see below). Under the CCV, scarcity does not eliminate co-creation. Even when buyers are abundant, each unit of value still needs a buyer-seller match, which limits the amount a seller gets. It is even possible for the per-seller payoff to approach 0, although this is in the extreme case that the sellers exit first. Finally, as $\rho \to 1$, the limiting interval for both per-buyer and per-seller payoffs is $[1 - \ln 2, \ln 2]$. This interval includes the case of equal division between buyers and sellers, which arises when buyers and sellers take turns in leaving.

In sum, the CCV ranges between highly asymmetric and equal divisions depending on how breakdowns unfold.

The core of the Glove Game gives 0 to each buyer and 1 to each seller, as soon as $B > S$. This is true regardless of how large B and S are, which seems like a knife-edge answer (as observed in Aumann 2, 2008). The Shapley value smooths out the core solution in finite-size games. However, for any case with $S/B \to \rho = 1 - \epsilon$, however small the imbalance $\epsilon > 0$, the per-buyer payoff goes to 0 and the per-seller payoff goes to 1, so that, in a sense, knife-edge behavior returns. The reason is that, under Shapley, as soon as sellers are on the short side, buyers are almost never pivotal in large random orders.

For the next examples, we consider situations with partially endogenous probabilities. We make three assumptions on the breakdown probabilities: (i) only partitions $\pi = \{\{i\}, S\backslash\{i\}\}$ where one player is left out receive positive weight; (ii) probabilities are non-increasing in losses, that is, if the loss when player j is left out is at least as large as when player i is left out, the probability of the first partition is at most equal to the probability of the second partition; and (iii) if the loss is 0 when some player i is left out, then only 0-loss partitions receive positive weight and the weights on these partitions are equal.

Example 9. *(**The Overcapacity Game**) The structure of this game is due to Postlewaite and Rosenthal (39, 1974). There are two sellers, each with two units to offer at cost 0. There are three buyers and each buyer is interested in one unit which they value at 1. The idea of the game is that there is aggregate but not individual overcapacity.*

The loss created at the root of the game when either a single buyer or seller is left out is 1. This is divided equally, so that each player receives 1/5. There are then two cases to consider.

(i) If a buyer is left out, the resulting subset contains two sellers and two buyers. At this point, an individual seller adds no value and so each seller is left out with probability 1/2. This leads to a subset consisting of one seller and two buyers.

(ii) If a seller is left out, the resulting subset contains one seller and three buyers. At this point, an individual buyer adds no value and so each buyer is left out with probability 1/3. This again leads to a subset containing one seller and two buyers.

Let p be the chance that any individual buyer is left out at the 1-seller, 2-buyer game, so that $1 - 2p$ is the probability the seller is left out. The expected loss is then $2p \times 1 + (1 - 2p) \times 2 = 2(1 - p)$ and is divided equally among the three players. If a buyer is left out, the resulting subset contains one seller and one buyer. The value of 1 is shared equally between them. If a seller is left out, there is 0 left to divide. The overall expected payoffs to each buyer and each seller in the game are therefore:

$$\phi_S = \frac{1}{5} + (\frac{1}{5} + 3 \cdot \frac{1}{5} \cdot \frac{1}{2})\big[\frac{2(1-p)}{3} + 2p \cdot \frac{1}{2}\big] = \frac{1}{5} + \frac{2+p}{6}, \tag{80}$$

$$\phi_B = \frac{1}{5} + (2 \cdot \frac{1}{5} \cdot \frac{2}{3} + 2 \cdot \frac{1}{5})\big[\frac{2(1-p)}{3} + p \cdot \frac{1}{2}\big] = \frac{1}{5} + \frac{4-p}{9}. \tag{81}$$

Since probabilities are non-increasing in losses, the parameter p must lie in the interval $[1/3, 1/2]$, from which:

$$0.589 \leq \phi_S \leq 0.617 \text{ and } 0.589 \leq \phi_B \leq 0.607. \tag{82}$$

The core of the Overcapacity Game gives each seller 0 and each buyer 1. Arguably, the core solution underestimates the position of a seller in the game, since, without a seller, the overall value shrinks from 3 to 2. It seems an extreme answer to award each seller 0. (The mechanism by which the core delivers this answer is that the buyers play the sellers off against each other for each unit of capacity.) Direct calculation shows that the Shapley value gives each seller $13/20 = 0.65$ and each buyer $17/30 = 0.567$. Arguably, this goes somewhat too far in the opposite direction in favoring the sellers. Under our assumptions, the CCV gives a strictly intermediate range of answers. Its solution set is closely clustered around an equal five-way division giving 3/5 to each of the five players, which looks more reasonable.

Example 10. *(**The Airport Problem**) Three airlines want to share the cost of building a runway. (The classic analysis is Littlechild and Owen 20, 1973. For a recent survey, see Thomson 54, 2024.) Airline a, with the smallest aircraft, needs a runway of length 1, airline b needs a runway of length 2, and airline c, with the largest aircraft, needs a runway of length 3. The construction cost is 1 per unit length of runway. In the absence of coordination, the total cost of three separate runways is $1+2+3=6$. Under coordination, the total cost falls to 3. This scenario naturally leads to the cooperative game:*

$$v(\{a,b,c\}) = 3,\ v(\{a,b\}) = v(\{a,c\}) = 1,\ v(\{b,c\}) = 2, \tag{83}$$

with $v(\{i\}) = 0$ for each $i = a, b, c$.

The Shapley value is obtained in a simple way. The cost of the first leg of the shared runway is divided equally among all three players, the cost of the second leg is divided equally among players b and c, and the cost of the third leg is fully borne by player c. This is an intuitive solution. Under it, player a saves 2/3 and pays 1/3. Nevertheless, we believe that the CCV analysis under the assumptions of this section suggests that there is a second intuitive solution.

Under this scenario, the worst case for player a is when breakdown leads with probability 1 to the partition where it is left out. In this case, we have $m(\{a,b,c\}) = 1$ and so player a gets 1/3 and therefore pays 2/3. The best case for player a is when all three partitions that leave out one player are equally likely. In this case, player a gets 8/9 and pays 1/9. This is also the Solidarity value. If the breakdown probabilities are loss-harmonic (Definition 9), then

$$p_N(\{\{a\},\{b,c\}\}) = \frac{1}{2},\ p_N(\{\{b\},\{a,c\}\}) = p_N(\{\{c\},\{a,b\}\}) = \frac{1}{4}, \tag{84}$$

from which $m(\{a,b,c\}) = 3/2$. In total, player a gets $1/3 \cdot 3/2 + 2 \cdot 1/4 \cdot 1/2 = 3/4$ and therefore pays 1/4. In sum, under the CCV assumptions of this section, player a pays an amount that varies from 1/9 to 2/3, with 1/4 as a salient but not inevitable answer.

The worst-case scenario for player a helps illuminate the difference between the Shapley and CCV analyses. Absent a three-way agreement, players b and c have the biggest incentive to form a coalition. When they do so, player a can still offer an additional saving of 1 on the first leg to players b and c. This is split three ways so that player a gets 1/3 and pays 2/3. This worst case for player a does not strike us as unreasonable. The Shapley value agrees with this logic as far as the second and third legs are concerned. However, the Shapley value treats the first leg (just like each of the second and third legs) as effectively a separate game, and the cost is divided equally, so that player a gets 2/3 and pays 1/3. But one can ask why players b and c would not extend their coalition to the first leg. If they do, player a might well get less than 2/3, more in line with the CCV.

Example 11. *(**A Voting Game**) Consider a voting game (following Shapley and Shubik 49, 1954) where there are four parties a, b, c, d of sizes 10, 20, 30, and 40, respectively. A strict majority, i.e., a size greater than 50, is required to win. We set the value of winning to 1. Under the CCV assumptions of this section, partitions with loss 0 are selected first, with equal probability. In the first round, that leads to an equal probability of any party being left out. If the resulting three-party coalition is $\{b,c,d\}$, then either party b or party c can be left out without loss. Party d and the remaining party b or c share 1 equally. The analysis of the other three-party coalitions is analogous. The overall expected payoffs are:*

$$\phi_a = \frac{1}{4}\big(0+0+0+\frac{1}{3}\big) = \frac{1}{12}, \tag{85}$$

$$\phi_b = \frac{1}{4}\big(\frac{1}{2}\cdot\frac{1}{2}+0+\frac{1}{2}+\frac{1}{3}\big) = \frac{13}{48}, \tag{86}$$

$$\phi_c = \frac{1}{4}\big(\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}+0+\frac{1}{3}\big) = \frac{13}{48}, \tag{87}$$

$$\phi_d = \frac{1}{4}\big(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+0\big) = \frac{3}{8}. \tag{88}$$

The Shapley-Shubik value for this game gives the parties 1/12, 1/4, 1/4, and 5/12, respectively. Party a gets the same under the two solutions, while parties b and c get a little more under the CCV and party d gets a little less. The breakdown probabilities corresponding to our assumptions temper the power of the largest party.

8 Hitting Probabilities and the Shapley Value

In this section, we show how to obtain the Shapley value as the CCV of an extended subset-refinement game. To do so, we extend the definition of a subset-refinement game by allowing each p_S to be a signed weight distribution on $\Pi(S)$ with total weight 1. We previewed a three-player example of this extension in Section 4. Here, we provide the general treatment, starting with the formal relationship and then giving some interpretations.

Theorem 4. *Fix a game $\mathcal{G} = (N, v, G, p)$. Suppose all hitting probabilities are equal to 1, that is, for all nonempty $T \subseteq S \subseteq N$, we have $\lambda_S(T) = 1$. Then, the CCV equals the Shapley value: $\phi_i(S) = \phi_i^{\mathrm{Sh}}(S)$ for all S.*

Proof. Since all hitting probabilities are 1, Equation 30 simplifies to:

$$\phi_i(S) = \sum_{i \in T \subseteq S} \frac{m(T)}{|T|} \tag{89}$$

Using Efficiency, we get from Equation 89:

$$v(S) = \sum_{i \in S} \phi_i(S) = \sum_{i \in S} \sum_{i \in T \subseteq S} \frac{m(T)}{|T|} = \sum_{T \subseteq S} \sum_{i \in T} \frac{m(T)}{|T|} = \sum_{T \subseteq S} m(T), \tag{90}$$

where we reorder indices in the double sum. Taking out the top term in the summation and rearranging,

$$m(S) = v(S) - \sum_{T \subsetneq S} m(T). \tag{91}$$

This is identical to the Harsanyi dividend recursion of Equation 32. Since $m(\{i\}) = v(\{i\}) = D(\{i\})$ for all i, it then follows that $m(S) = D(S)$ for all $S \subseteq N$. From this, the CCV given by Equation 89 and the Harsanyi form of the Shapley value given by Equation 31 are equal, completing the proof. □

Since the CCV coincides with the Shapley value here, a Shapley dummy player must receive $v(\{i\})$ under the CCV. We can go further.

Corollary to Theorem 4. *Suppose all hitting probabilities are equal to 1. Then, if player i is a Shapley dummy, it is a CCV dummy.*

Proof. The proof of Theorem 4 established that $m(S) = D(S)$ for all $S \subseteq N$. If player i is a Shapley dummy, then $D(U) = 0$ for all nonsingleton $U \subseteq N$ with $i \in U$. From this, we get $\lambda_N(U)\, m(U) = \lambda_N(U)\, D(U) = 0$ for such U. □

A necessary and sufficient condition for the CCV to coincide with the Shapley value at every subset for every characteristic function is that all hitting probabilities are equal to 1.

Theorem 5. *Fix a player set N, a subset-refinement graph G, and a family of breakdown probabilities p. Suppose that, for every characteristic function v, the CCV equals the Shapley value at every subset, that is: $\phi_i(S) = \phi_i^{\mathrm{Sh}}(S)$ for all nonempty $S \subseteq N$ and all $i \in S$. Then for nonempty $T \subseteq S \subseteq N$, we have $\lambda_S(T) = 1$.*

The next task is to identify circumstances under which all hitting probabilities equal 1. In the three-player case (Figure 1), this requirement implies $\alpha = \beta = \gamma = 1$, which forces $\delta = -2$, so that we get a signed breakdown weight. It follows that for any game with three or more players, the unit hitting probability condition cannot be satisfied with ordinary (unsigned) probabilities. We see that $(1, 1, 1, -2)$ is the unique measure satisfying the condition in the three-player case. Moreover, it is an instance of the upper-interval Möbius distribution (Stanley 53, 2012, Section 3.10), whose general formula is:

$$p_S^{\mathrm{Mo}}(\pi) = (-1)^{|\pi|}\, (|\pi| - 1)!, \tag{92}$$

for each nontrivial partition $\pi \in \Pi(S)$. This is the unique distribution where the probability of a partition depends only on the number of cells in the partition and all hitting probabilities are equal to 1 (Rota 42, 1964). It is a canonical choice of signed breakdown probabilities but, for games with four or more players, it is not the unique such distribution.

Another choice with unit hitting probabilities is what we call the One-Big-Block (1BB) distribution:

$$p_S^{\mathrm{1BB}}(\pi) = \begin{cases} (-1)^{|S\setminus B|+1} & \text{if } \pi = \{B\} \cup \{\{j\} : j \in S\setminus B\},\ \ 2 \le |B| \le |S|-1; \\ (-1)^{|S|}(|S|-1) & \text{if } \pi \text{ is the discrete partition of } S; \\ 0 & \text{otherwise.} \end{cases} \tag{93}$$

Here, for each subset S, only partitions with at most one nonsingleton cell receive nonzero weight. Proposition E.1 checks that 1BB correctly sums to 1. We also confirm the unit-hitting property, which we will use in Section 9.

Proposition 5. *Under the 1BB weights, all hitting probabilities are* 1.

There is more than one way to interpret signed breakdown weights as an extension to our theory. One way is purely calculational. We compute the CCV via the basic recursion formula of Equation 23, simply using signed weights $p_S(\pi)$ where they appear. A second way is to go back to the playable upward process described at the end of Section 5. We sample upward runs under probabilities proportional to the absolute values of their weights, attach to a player's payoff on each sampled run the sign of the original weight on the run, and multiply the resulting average by the total absolute weight. This procedure gives an unbiased statistical implementation of the same answer as direct calculation (Abramsky and Brandenburger 1, 2014), while retaining the playability of the upward process.

There is a dual to the upper-interval Möbius distribution, namely, the lower-interval Möbius distribution on the partition lattice (Stanley 53, Section 3.10). Renormalized to exclude the trivial partition $\{S\}$, it is given by:

$$p_S^{\mathrm{mo}}(\pi) = \frac{(-1)^{|\pi|}}{(|S|-1)!} \prod_{U\in\pi} (|U|-1)!. \tag{94}$$

This distribution has non-negative hitting probabilities (see Lemma E.2) and also leads to a closed-form expression for the CCV.

Proposition 6. *Fix a game $\mathcal{G} = (N, v, G, p)$ where the family of signed breakdown weights p consists of the lower-interval Möbius distributions. The CCV is given by:*

$$\phi_i^{\mathrm{mo}}(N) = v(\{i\}) + \sum_{\substack{i\in T\subseteq N \\ |T|\ge 2}} \frac{1}{\binom{|N|-1}{|T|-1}} \cdot \frac{1}{|T|}\big[v(T) - \frac{1}{|T|-1}\sum_{j\in T} v(T\setminus\{j\})\big]. \tag{95}$$

To get a sense of how this version of the CCV with signed probabilities works, we revisit two examples from Section 7.

Example 8 contd. ***(The Glove Game)*** *Suppose a subset T contains b buyers and s sellers. Under the lower-Möbius weighting, we find $m(T) = 0$ if $s \neq b$. To see this, assume without loss of generality that $b > s$. This implies $v(T) = \min\{b, s\} = s$. Removing any buyer leaves the value unchanged at s, while removing any seller lowers the value to $s-1$. We therefore have:*

$$v(T) - \frac{1}{|T|-1}\sum_{j\in T} v(T\setminus\{j\}) = s - \frac{bs + s(s-1)}{b+s-1} = 0. \tag{96}$$

That is, no player gets any payoff until we reach a balanced subset with $s = b$. At this point, the two sides are fully symmetric (alternatively put, the labels b and s are arbitrary). Therefore, at any balanced subset, the remaining buyers and sellers split the total equally. Adding up over all balanced subsets, we conclude that the total payoff to buyers equals the total payoff to sellers. Now go back to the top of the graph. The total value is S, so, by Efficiency, each buyer gets $1/B \times S/2 = S/2B$ and each seller gets $1/S \times S/2 = 1/2$.

The striking answer is that the lower-Möbius CCV splits the total value equally between the two sides of the market, giving each side $S/2$. This amount is then split equally among participants on that side. Contrary to the core and Shapley value answers we saw earlier, the payoffs are independent of the total of players if the ratio $S/B = \rho$ is fixed.

Also notable is that the lower-interval Möbius answer coincides with the CCV of the Glove Game if we make assumptions (i), (ii), and (iii) of Section 7 on ordinary non-negative breakdown probabilities. To see this, first observe that away from balance, zero-loss exits from the long side are selected uniformly until balance is restored. At balance, all one-player exits have a loss of 1, so they are selected uniformly. This alternation gives each player present at a balanced stage expected value 1/2. With $B \ge S$, the

probability that a buyer survives the initial thinning is S/B and the survival probability for a seller is 1. Putting this together, each buyer gets an expected payoff of $S/2B$ and each seller an expected payoff of $1/2$. This mechanism and the lower-interval Möbius calculation yield the same answer because both reduce the calculation to equal division of the value created at balanced subsets. Under the Section 7 style analysis, this reduction is reached via assumptions on probabilistic breakdown. Under the lower-interval Möbius analysis, the conclusion is reached via signed cancellation. The endpoint is the same. Finally, note that this example provides an instance where the lower-interval Möbius CCV, although based on signed weights, lies in the non-negative CCV polytope.

The formula for the lower-interval Möbius CCV in Equation 95 bears a strong resemblance to the general formula one obtains for the Solidarity value. Recall from Section 4 that this coincides with the CCV when zero-normalized and restricted to equiprobable one-player exits, so the formula is:

$$\phi_i^{\mathrm{So}}(N) = \sum_{i \in T \subseteq N} \frac{1}{\binom{|N|}{|T|}} \cdot \frac{1}{|T|} \big[v(T) - \frac{1}{|T|} \sum_{j \in T} v(T \backslash \{j\}) \big]. \tag{97}$$

Although close in form, the two answers will in general be different. For example, in the Glove Game with $B = 2$ and $S = 1$, the lower Möbius solution is $1/4$ to each buyer and $1/2$ to the seller, while the Solidarity value gives $5/18$ to each buyer and $4/9$ to the seller.

Example 10 contd. ***(The Airport Problem)*** *Applying Equation 95, each two-player subset T containing player i contributes:*

$$\frac{1}{\binom{2}{1}} \cdot \frac{1}{2} v(T) = \frac{1}{4} v(T), \tag{98}$$

while the grand coalition contributes to each player:

$$\frac{1}{3} \big[v(\{a,b,c\}) - \frac{1}{2} \big(v(\{a,b\}) + v(\{a,c\}) + v(\{b,c\}) \big) \big] = \frac{1}{3}. \tag{99}$$

The lower Möbius solution is:

$$\phi_a^{\mathrm{mo}} = \frac{1}{4} v(\{a,b\}) + \frac{1}{4} v(\{a,c\}) + \frac{1}{3} = \frac{1}{4} + \frac{1}{4} + \frac{1}{3} = \frac{5}{6}, \tag{100}$$

$$\phi_b^{\mathrm{mo}} = \frac{1}{4} v(\{a,b\}) + \frac{1}{4} v(\{b,c\}) + \frac{1}{3} = \frac{1}{4} + \frac{1}{2} + \frac{1}{3} = \frac{13}{12}, \tag{101}$$

$$\phi_c^{\mathrm{mo}} = \frac{1}{4} v(\{a,c\}) + \frac{1}{4} v(\{b,c\}) + \frac{1}{3} = \frac{1}{4} + \frac{1}{2} + \frac{1}{3} = \frac{13}{12}. \tag{102}$$

Equivalently, airline a pays $1/6$, airline b pays $11/12$, and airline c pays $23/12$.

Relative to Shapley, the lower Möbius answer shifts cost away from the smallest airline and toward the two larger airlines. Comparing with the CCV analysis in Section 7, the lower Möbius payment of $1/6$ for player a lies inside the range $[1/9, 2/3]$ we identified and is close to the salient loss-harmonic CCV payment of $1/4$. Overall, the lower Möbius solution concept offers another way of moving away from the Shapley equal sharing of the first leg and towards an answer in which the larger airlines bear more of the common cost.

We end this section with a third interpretation of signed breakdown probabilities. Go back to the three-player case and consider the following scenario. Player a believes with probability 1 that, in the event of a breakdown, it will be paired with another player. This suggests we set $\beta + \gamma = 1$. Player b likewise believes that, in the event of a breakdown, it will be paired with another player. So, we set $\alpha + \gamma = 1$. With the analogous belief for player c, we set $\alpha + \beta = 1$. We can sum up by saying that the three players each take an "optimistic" view of what happens after breakdown. But, taken together, these three views imply $2(\alpha + \beta + \gamma) = 3$, which is impossible if probabilities are nonnegative. If we allow signed probabilities, then we can find a solution by setting, say, $\alpha = \beta = \gamma = 1/2$ and $\delta = -1/2$, which is, in fact, the lower-interval Möbius distribution for three players. Thus, the introduction of signed breakdown weights allows us to describe a symmetric scenario—in which each player believes it will be in a surviving coalition after breakdown—that cannot be described in the unsigned case. This observation may point to another kind of symmetry principle (different from our Axiom 2) that yields a signed-weight representation theorem. We leave a careful treatment of this idea to future work.

9 Externalities

The treatment of externalities in cooperative game theory goes back to the partition-function approach of Lucas (21, 1963) and Thrall and Lucas (55, 1963). Externalities are modeled by allowing the value of a coalition to depend on an entire partition of the player set. Much subsequent work has focused on extending the Shapley value to this richer setting, beginning with Myerson (33, 1977) and continuing with Bolger (5, 1989), Macho-Stadler, Pérez-Castrillo, and Wettstein (23, 2007), de Clippel and Serrano (8, 2008), McQuillin (30, 2009), Dutta, Ehlers, and Kar (10, 2010), Skibski and Michalak (51, 2020), and Navarro, Rémila, and Solal (35, 2025), among others. Maskin (28, 2003) explores the possibility that externalities may work against formation of the grand coalition. (We return to non-formation of coalitions later.) A separate strand of work includes Ray and Vohra (41, 1999), who simultaneously endogenize the partition that forms and the division of value within coalitions as the outcome of an associated noncooperative bargaining game; see Ray (40, 2007) for a synthesis of subsequent developments in this direction.

The CCV framework suggests a natural way to model the effect of externalities. A subset-refinement graph describes the sequence of partition-subset pairs that arise on the path to a given subset. Allowing the value of a subset to depend on this history provides a simple extension of the CCV to games with externalities. The only modification required is to make the definition of the characteristic function context-dependent. The recursive definition of the CCV and its representation in terms of hitting probabilities then extend directly.

To preview our approach, return to the three-player game as depicted in Figure 1. Without externalities, the standalone value $v(\{a\})$ for player a is the same at all four occurrences of the singleton set $\{a\}$. With externalities, the literature typically distinguishes two values:

$$v(\{a\} \,|\, \{b,c\}) \text{ and } v(\{a\} \,|\, \{b\},\{c\}). \tag{103}$$

The first value is the standalone value of player a when b and c are together, while the second value is the standalone value of player a when b and c are separate. Our general concept of externality is broader and allows values to differ across (partial) runs that reach each occurrence of a given subset. Thus, in the three-player example, the standalone value of a may involve up to four different quantities:

$$v(\{a\} \,|\, \alpha),\ v(\{a\} \,|\, \beta),\ v(\{a\} \,|\, \gamma),\ v(\{a\} \,|\, \delta). \tag{104}$$

Here, $v(\{a\} \,|\, \alpha)$ is the standalone value of a on the left-most branch in Figure 1 and is where b and c are together, and the remaining three quantities are when b and c are separate—but may differ because the order of exit of b and c differs.

Our richer scenario might arise if, for example, every breakdown entails a cost κ. This cost is incurred twice on the β and γ branches, but only once on the δ branch, and we might then set:

$$v(\{a\} \,|\, \beta) = v(\{a\} \,|\, \gamma) = v(\{a\} \,|\, \delta) - \kappa. \tag{105}$$

The existing literature generally assumes path independence (Dutta, Ehlers, and Kar (10, 2010)), which says that all that matters is the ambient partition where the subset in question arises. In the three-player case, this allows only:

$$v(\{a\} \,|\, \alpha) \neq v(\{a\} \,|\, \beta) = v(\{a\} \,|\, \gamma) = v(\{a\} \,|\, \delta). \tag{106}$$

Example 12. ***(A Perturbed Unanimity Game)*** *Let $N = \{a,b,c\}$ and begin with the unanimity game where $v(N) = 1$ and every strict subset of N has value 0. We consider two different externality perturbations:*

$$v(\{a\} \,|\, \{b\},\{c\}) = \eta \ \textit{and}\ v(\{a\} \,|\, \{b,c\}) = 0, \tag{107}$$

$$v(\{a\} \,|\, \{b\},\{c\}) = 0 \ \textit{and}\ v(\{a\} \,|\, \{b,c\}) = \eta, \tag{108}$$

for some $\eta \neq 0$. The externality distinguishes only whether b and c are separate ("sep") or together ("tog"), so this is a case of path independence.

Assuming equal probabilities $\alpha = \beta = \gamma$, a straightforward calculation yields:

$$\phi_a^{\text{sep}} = \frac{1}{3} + \frac{(2-3\alpha)\eta}{3} \ \textit{and}\ \phi_a^{\text{tog}} = \frac{1}{3} + \frac{2\alpha\eta}{3}. \tag{109}$$

Therefore, for a positive externality η, player a does better under sep than tog whenever $\alpha < 2/5$, which is always the case under our equal-probability condition (which implies $\alpha \leq 1/3$). The reverse is true for a negative externality $\eta < 0$. This result does not depend on the size of η.

It seems intuitive that player a benefits more from a positive externality when players b and c are not acting together. Less immediately intuitive is what happens if we choose the signed breakdown probabilities

$\alpha = \beta = \gamma = 1, \delta = -2$, and $\eta = 1$, which yields $\phi_a^{\text{sep}} = 0$. At first blush, one would think that player a would do better than the 1/3 it gets in the game without the positive externality ($\eta = 0$).

The decomposition of games developed in Section 5 helps clarify what is happening. Decompose the game into the eight basic deterministic games that are not identically 0. On the α-branch, the only nonzero loss is 1 at $\{a, b, c\}$. The associated basic deterministic game is the ordinary unanimity game and gives each player 1/3. On the β-branch, there are three nonzero local losses:

$$l(\{a, b, c\}) = 1,\ l(\{a, c\}) = -1,\ l(\{a\}) = 1. \tag{110}$$

The corresponding basic deterministic games give player a payoffs of $1/3, -1/2$, and 1, respectively. By symmetry, player a also receives $1/3, -1/2$, and 1 on the γ-branch. On the δ-branch, the nonzero value arises at the singleton $\{a\}$, and so player a receives 1. Applying the signed weights, we get:

$$\phi_a^{\text{sep}} = (\frac{1}{3} + \frac{1}{3} + \frac{1}{3}) + (-\frac{1}{2} - \frac{1}{2}) + (1 + 1 - 2). \tag{111}$$

The first term in brackets refers to the grand coalition, the second term to the two pairs including a, and the third term to the $\{a\}$-singletons. We see that the negative weight on the δ-branch cancels the positive contributions from the β- and γ-branches. The local losses at the grand coalition give player a a total of 1 across the α-, β-, and γ-branches. However, on the latter two branches, the externality is not (yet) realized at the intermediate pairs, which produces local losses of -1, half of which are incurred by player a. This cancels what a receives from the grand coalition. We see that a positive externality does not enter the decomposition as a one-time benefit but, rather, creates counteracting intermediate losses. The net effect in this example is to cancel exactly the gain to player a. In Example 13 to come, we will see this effect at work again.

A notable feature of this game is the appearance of negative local losses at the subsets $\{a, c\}$ and $\{a, b\}$. This raises the question of what happens if we allow the players not to agree (to equal division) in such cases and instead let the subset actually break down. We comment on this possible extension of our theory in Section 11.d.

9.1 General Framework

We now present our general framework for subset-refinement games with externalities. The key idea is that the value of a coalition may depend not only on the coalition itself, but also on the sequence of prior breakdowns through which that coalition is reached. Fix a player set N and the subset-refinement graph $\mathcal{G}$ defined in Section 3. Recall from Section 5 that a run r is a complete specification of deterministic breakdowns in the graph, including breakdowns at subsets that are not reached from the top under that specification. We quotient out the precluded choices by treating all runs that reach a given occurrence of a subset T as equivalent.[8] An occurrence of T is a particular path in the subset-refinement graph from N to T, including the partitions encountered along that path.

Definition 10. *Fix an occurrence of $T \subseteq N$. A **partial run** r_T to T is the equivalence class of runs that reach that occurrence of T.*

Definition 11. *A **context** is a pair (T, r_T) where r_T is a partial run to T.*

Formally, the top context can be written as $(N, \emptyset)$, but we will usually drop the $\emptyset$ symbol. Write $v(T, r_T)$ for the value that a context-dependent characteristic function gives coalition T in context (T, r_T). Next, suppose that T breaks down into a partition $\pi \in \Pi(T)$. This extends the partial run by one step. For each cell $U \in \pi$, let $r_{T,U}$ denote the induced partial run to U, so that the context associated with U is $(U, r_{T,U})$. We can now write down the local loss associated with the breakdown to π:

$$l_\pi(T, r_T) = v(T, r_T) - \sum_{U \in \pi} v(U, r_{T,U}). \tag{112}$$

In this section, we allow contextual characteristic functions but retain context-independent breakdown probabilities. In Section 10.2, we will need both forms of contextuality. For now, we write the expected local loss as:

$$m(T, r_T) = \sum_{\pi \in \Pi(T)} p_T(\pi)\, l_\pi(T, r_T), \tag{113}$$

with $m(\{i\}, r_{\{i\}}) = v(\{i\}, r_{\{i\}})$. We now define the CCV with externalities.

[8] Consider the breakdown $\{a, b, c\} \to \{a\} \,|\, \{b, c\}$. The partial run does not indicate the subsequent breakdown of the pair $\{b, c\}$. Conceptually, players b and c are assumed to have reached an agreement. This makes sense when we remember that breakdowns are a counterfactual—not actual—process that determines the allocation of value.

Definition 12. *Fix a subset-refinement game $\mathcal{G} = (N, v, G, p)$ with externalities. The* ***co-creation value with externalities (Extended CCV)*** *of $\mathcal{G}$, denoted as before by $\phi^{\mathcal{G}}$, associates to each nonsingleton occurrence (S, r_S) and player $i \in S$, the value $\phi_i^{\mathcal{G}}$ given by the one-step recursion:*

$$\phi_i^{\mathcal{G}}(S, r_S) = \frac{m(S, r_S)}{|S|} + \sum_{\pi \in \Pi(S)} p_S(\pi)\, \phi_i(\pi_i, r_{S,\pi_i}), \tag{114}$$

with initial condition $\phi_i(\{i\}, r_{\{i\}}) = v(\{i\}, r_{\{i\}})$.

This recursive definition has the same form as the original Definition 3, differing only in that the values are indexed by context. An analogous induction argument to that in the proof of Proposition 1 establishes the extended CCV is efficient at every context.

Proposition 7. *Efficiency holds at each context (S, r_S): $\sum_{i \in S} \phi_i(S, r_S) = v(S, r_S)$.*

The analog to the proof of Proposition 2 shows that the extended CCV can be written in hitting-probability form. Fix a context (S, r_S) and let $\mathcal{C}(S, r_S)$ denote the family of contexts reachable from (S, r_S). For each $(T, r_T) \in \mathcal{C}(S, r_S)$, let $\lambda_{(S,r_S)}(T, r_T)$ denote the hitting probability of (T, r_T) from (S, r_S). For $i \in S$, define:

$$\phi_i^*(S, r_S) = \sum_{\substack{(T, r_T) \in \mathcal{C}(S, r_S) \\ i \in T}} \lambda_{(S,r_S)}(T, r_T)\, \frac{m(T, r_T)}{|T|}. \tag{115}$$

Proposition 8. *For each context (S, r_S) and player $i \in S$, we have $\phi_i^*(S, r_S) = \phi_i(S, r_S)$.*

9.2 The Shapley Value with Externalities

The question of how to extend the Shapley value to games with externalities has remained a central challenge in cooperative game theory. Myerson (33, 1977) provides a pioneering axiomatic approach. Bolger (5, 1989) evaluates marginal contributions for each organization of players outside a given coalition. Macho-Stadler, Pérez-Castrillo, and Wettstein (23, 2007) average the value of a coalition over possible partitions of outside players and then apply the usual Shapley value. De Clippel and Serrano (8, 2008) derive bounds by replacing the externality game with upper and lower benchmark games without externalities. Dutta, Ehlers, and Kar (10, 2010) specify benchmark partitions of outside players and interpolate between them. It seems fair to say a definitive extension of the Shapley value with externalities has yet to be found (McQuillin 30, 2009). Here, we examine how the extended CCV may help shed light on this question. We begin with an example from de Clippel and Serrano (8, 2008, Example 8), which enables a detailed comparison of approaches.

Example 13. ***(A Large-N Externality Game)*** *Let $|N| = 101$ and set $v(N) = v(\{a_1\}|\{a_2\}, \ldots, \{a_{101}\}) = 1$, with the values of all other contexts equal to 0. In words, there is a positive externality for player a_1 when all other players are separate, and not otherwise. Table 1 shows the solutions for this game under different proposals, drawing on de Clippel and Serrano (8, 2008).*

Solution Concept	**Player a_1**	**Players $a_2, \ldots, a_{101}$**
Upper Möbius / Myerson	$\frac{1 - 99!}{101}$	$\frac{1}{101} + \frac{99!}{10,100}$
One-Big-Block	0	$\frac{1}{100}$
McQuillin	$\frac{1}{101}$	$\frac{1}{101}$
Lower Möbius / Bolger / MPW	$\frac{1}{101} + \frac{1}{101!}$	$\frac{1}{101} - \frac{1}{100 \times 101!}$
Single-Exit Uniform / dCS / DEK	$\frac{2}{101}$	$\frac{99}{10,100}$

Table 1: Solutions to the Large-N Externality Game

We start with the upper-Möbius CCV, which takes a very simple form in this game. (See Proposition F.1 in Appendix F for details.) After cancellation across partial runs, only two types of contribution from the externality survive. The first is when the grand coalition breaks into singletons. The second is when one outsider first exits and the remaining $(|N|-1)$-player cell containing a_1 then breaks into singletons. Paths on which two or more outsiders have already been separated cancel in aggregate. The net externality contribution to a_1 is $(-1)^{|N|}(|N|-2)!/|N|$. Adding the $1/|N|$ contribution from the grand coalition yields the entries in the top row of Table 1.

Myerson (33, 1977) proposes axioms for the Shapley value in the presence of externalities that yield a unique solution. This is the same as our solution when the externalities are path-independent and we use upper-Möbius weights. The present example demonstrates the instability of the Myerson value. The payoff to player a_1 is approximately -10^{154}, a highly implausible answer. Moreover, for $|N| = 102$, the payoff to a_1 swings to an implausibly large positive number.

Theorem 4 showed that there is an unambiguous route from the CCV to the Shapley value when there are no externalities. Any family of breakdown probability measures that makes all hitting probabilities equal to 1 yields the Shapley value. This is no longer true once externalities are allowed, which gives us an "escape route" from the extreme and unstable answer just found.[9]

In this vein, consider the One-Big-Block (1BB) solution in the second row of the table. For the externality component, the aggregate signed weight of all partial runs reaching a context in which every player is a singleton is 0, so the direct singleton value of the externality cancels. What remains is the effect of the local losses along the way. At a context where every player outside the cell containing a_1 is already a singleton, the externality component gives a_1 a net payoff of $1/2$ when that cell has two players, and $-1/s$ when it has $s \geq 3$ players. (See Proposition F.2.) Therefore, at the top, the externality gives $-1/|N|$ to a_1, exactly canceling its $1/|N|$ share of the grand-coalition component and leaving $a_1$with a net payoff of 0. This is the same mechanism as in the sep externality of Example 12. The signed weights cancel the direct singleton benefit from the externality, and the negative local losses cancel the benefiting player's share of the value of the grand coalition.

McQuillin (30, 2009) constructs a recursive Shapley-type extension, which, together with his other axioms, implies that each coalition is evaluated only under the coarsest partition containing it. In the current example, this effectively eliminates the all-singleton externality, leaving an ordinary unanimity game, so that each player receives $1/|N|$. This is the third row of the table. The same answer can be obtained via the CCV with ordinary breakdown probabilities where every subset divides into two equal or nearest-equal cells. The initial breakdown is then to cells with 50 and 51 players, and we cannot reach the all-singleton context where the externality arises.

The Bolger solution (5, 1989) satisfies additivity, so we can again consider separately the value of the grand coalition and the externality. As before, the first gives every player $1/|N|$. Bolger evaluates the externality through the marginal effects of an outside player joining a_1, combined using weights determined by his axioms. In the example, this produces an extremely small reallocation where a_1 receives an additional $1/|N|!$, given up equally by the other players. This and the next two solutions are the entries in the fourth row of the table. The Macho-Stadler, Pérez-Castrillo, and Wettstein (23, 2007, henceforth MPW) solution of a game with externalities works by considering, for each subset of players, a uniform distribution on permutations of the outside players and, for each draw, taking the cycle decomposition. This yields an average value for the subset in question, and MPW then calculate the ordinary Shapley value for this averaged characteristic-function game. In the present example, the externality only arises under the identity permutation of players $a_2, \ldots, a_{101}$. It is therefore averaged with weight $1/(|N|-1)!$. The resulting averaged game has $v(N) = 1$ and $v(\{a_1\}) = 1/(|N|-1)!$, with all other subsets receiving a value of 0, and MPW calculate its Shapley value.

The lower-Möbius solution agrees with the Bolger and MPW in this game (although not in general). The grand coalition gives $1/|N|$ to player a_1. For the externality, the aggregate continuations from subsequent contexts cancel. The complete breakdown at the top has weight $-1/(|N|-1)!$ and therefore contributes $1/|N|!$ to a_1. (See Proposition F.3.) This produces the same effect from the externality as under the Bolger and MPW solutions—and a solution that is, to any plausible degree of precision, again equal division among all players.

The "externality-free" value of de Clippel and Serrano (8, 2008) yields the fifth row of the table and is the solution most favorable to player a_1 (excluding upper Möbius for $N = 102$). Their construction replaces the game with externalities by the characteristic function $v^*(S) = v(S \,|\, \{S\} \cup \{\{j\} : j \in N \backslash S\})$, so that, in the present case, $v^*(N) = v^*(\{a_1\}) = 1$, and all other subsets receive value 0. The usual Shapley value then gives player a_1 the grand-coalition share of $1/|N|$ with an additional $1/|N|$ from the singleton value, for a

[9] With three players, there is a unique family of breakdown probabilities with unit hitting probabilities. In the notation of Figure 1, this is $\alpha = \beta = \gamma = 1, \delta = -2$. Therefore, all unit-hitting probability solutions coincide. With four players, all exchangeable unit-hitting breakdown families are affine combinations of the upper-interval Möbius distribution and the 1BB distribution of Equation 93. With five or more players, the number of degrees of freedom grows rapidly.

total of $2/|N|$. This is the de Clippel and Serrano upper-endpoint payoff to player a_1. Their lower endpoint is equal division of $1/|N|$ to every player. The Dutta, Ehlers, and Kar (10, 2010) solution concept computes Shapley-style marginal contributions by supposing that, after a player leaves a coalition, the player remains alone or joins one of the outside coalitions, with a parameter controlling the respective probabilities. In the current game, one of their parametrized solutions, which they call the "sing" solution, gives player a_1 exactly $2/|N|$. (See Example 3 in Dutta, Ehlers, and Kar 10, 2010.)

The CCV solution also yields the fifth-row solution, in the case that all breakdowns involve single-player exits and all exits from a given subset are equally likely. In the present game, this means that player a_1 will be one of the last two remaining players with probability $2/|N|$. At that point, the two players split the local loss of -1 on breakdown, and then a_1 gets the externality of 1. The net to a_1 is $-1/2+1=1/2$. The externality payoff is therefore $2/|N| \times 1/2 = 1/|N|$. Adding the $1/|N|$ share at the top, we find that a_1 gets $2/|N|$.

We see that the (extended) CCV places all of these answers within a single framework. Holding the characteristic function fixed, each row is generated by a different specification of the breakdown weights. Under the CCV logic, the weights simultaneously determine which partitions enter the calculation and—together with equal division of each local loss—how partitions lead to payoffs. Thus, the model of partition formation and the rule for dividing value are not chosen independently, which we think offers a clean internal structure that applies equally to games with or without externalities. Overall, the wide variation in player a_1's payoff across the rows in Table 1 can be read as the transparent consequence of different assumptions about how breakdown occurs, rather than as coming from very different analyses.

The point of our approach to games with externalities is not to claim a single definitive extension of the Shapley value, but to suggest a method of analysis. The current example shows that investigating the CCV as the breakdown probabilities vary offers a unified and, we believe, insightful analysis of the game. We leave to further work the investigation of other salient games in the same fashion.

10 Extensions

In this section, we first extend our theory to the NTU case. We next provide two additional axiomatizations—one in terms of consistency (Hart and Mas-Colell 14, 1989) and the other in terms of monotonicity (Young 58, 1985). We then sketch a non-cooperative route to the CCV and, finally, provide a regression-based derivation of the CCV.

10.1 The NTU Case

The CCV provides a way to endogenize disagreement points when extending NTU bargaining solutions to n-player games. Typically, the disagreement values associated with a full breakdown are given exogenously. But that leaves the disagreement points associated with partial breakdowns unspecified. For example, if $\{a,b,c\}$ breaks down into $\{\{a\},\{b,c\}\}$, we will likely want the disagreement values for b and c to be different from those under the complete breakdown $\{\{a\},\{b\},\{c\}\}$. The CCV's recursive framework allows us to endogenize the disagreement points associated with partial breakdowns.

As an example, consider the Nash bargaining solution (Nash 34, 1950; Zeuthen 59, 1930), which, for a given disagreement point d, maximizes the product:

$$\prod_{i=1}^{n}(u_i - \mathsf{d}_i), \tag{116}$$

over the individually rational portion of a concave utility frontier. This is appropriate if disagreement always leads to full breakdown. The CCV approach extends the n-player Nash solution to allow for partial breakdowns.

In the 3-player example of Figure 1, begin with the left-hand partition $\{\{a\},\{b,c\}\}$. Write the singleton values as $u_i^*(\{i\}) = \mathsf{d}_i$, for $i = a,b,c$. Then, player a receives d_a, while the continuation payoffs for b and c are found by maximizing $(u_b - \mathsf{d}_b)(u_c - \mathsf{d}_c)$ over the relevant utility frontier. Write the resulting values as $u_b^*(\{b,c\})$ and $u_c^*(\{b,c\})$. Performing the parallel calculations for the other three partitions of the grand coalition gives a continuation-payoff vector for each of the four ways the grand coalition can break down. Averaging these vectors according to the breakdown probabilities produces a new disagreement point, which we write as $\mathsf{d}_i(\{a,b,c\})$ for $i = a,b,c$. With the usual NTU assumptions of superadditivity, the solution to the three-player game is then obtained by maximizing the product:

$$\big(u_a - \mathsf{d}_a(\{a,b,c\})\big)\big(u_b - \mathsf{d}_b(\{a,b,c\})\big)\big(u_c - \mathsf{d}_c(\{a,b,c\})\big), \tag{117}$$

over the individually rational portion of the utility frontier. The same procedure applies to a general n-player game. Start with the singletons and work inductively up the subset-refinement graph.

In the TU case, this NTU construction reduces to our original CCV. This follows from the well-known fact that, in the superadditive TU case, maximizing the product in Equation 116 over $u_i \geq \mathsf{d}_i$, subject to $\sum_{i \in S} u_i \leq v(S)$, yields equal gains relative to disagreement:

$$u_i - \mathsf{d}_i = \frac{v(S) - \sum_{j \in S} \mathsf{d}_j}{|S|}. \tag{118}$$

This is exactly the expression used in the CCV recursion (Equation 23), with d_j interpreted as player j's expected continuation payoff following breakdown.

Our NTU approach is modular. In place of the Nash bargaining solution, we could employ the Kalai and Smorodinsky (16, 1975) solution. The recursive procedure provides the endogenous disagreement points that are otherwise missing when two-player NTU solutions—where disagreement points are naturally exogenous—are extended to n-player games.

Combining, say, the Nash solution with a unit-hitting family of signed breakdown weights, we could position our NTU proposal in the literature on NTU extensions of the Shapley value (Harsanyi 13, 1963; Shapley 47, 1969; Maschler and Owen 26, 1989, 27, 1992; McLean 29, 2002). Notice that, unlike in the TU case, where all unit-hitting families yield the (TU) Shapley value (Theorem 4), this direction in the NTU case will yield a family of proposals for an NTU Shapley value, according to whether we choose the upper Möbius, 1BB, or another signed distribution.

Note that the NTU application covers games where the values of subsets are monetary but players are risk averse. The recursion is applied to the utilities of the monetary allocations. When breakdown probabilities are nonnegative, continuation values are then ordinary expected utilities.

10.2 Consistency

Hart and Mas-Colell (14, 1989) define a notion of consistency (Aumann 2, 2008, pp.34-35) for a cooperative game. Fix a solution concept ψ defined on all games (N, v). Given a subset $T \subseteq N$, the reduced game relative to T is (T, v^{-T^c}), where, for $S \subseteq T$:

$$v^{-T^c}(S) = v(S \cup T^c) - \sum_{i \in T^c} \psi_i(S \cup T^c). \tag{119}$$

In words, the value of S in the reduced game is the value available when S is together with the deleted subset T^c, minus the payoffs assigned to the deleted players at the subset $S \cup T^c$. Hart and Mas-Colell (14, 1989) call a solution concept ψ consistent if for each player $j \in T$:

$$\psi_j^{-T^c}(T) = \psi_j(N), \tag{120}$$

that is, each player in the reduced game gets the same as in the original game. Theorem B in Hart and Mas-Colell (14, 1989) establishes that the Shapley value is the unique consistent solution concept that divides the surplus equally in two-player games.

In the Hart and Mas-Colell setting, the reduced value of a subset is determined by that subset alone. In our framework, deleting a player can cause the same surviving subset to inherit different values and breakdown probabilities along different partial runs, as the following example shows.

Example 14. *Let $N = \{a, b, c, d, e, f\}$, where we will delete player f. Consider the characteristic function given by $v(\{a, b, c, f\}) = v(\{b, c\}) = 1$, with $v(S) = 0$ for every other subset S. At the top, set:*

$$p_N(\{\{a, b, c\}, \{d, e, f\}\}) = p_N(\{\{a, b, c, f\}, \{d\}, \{e\}\}) = \frac{1}{2}. \tag{121}$$

We specify the following subsequent (deterministic) breakdowns:

$$\{a, b, c\} \to \{a\} \,|\, \{b, c\} \text{ and } \{a, b, c, f\} \to \{a, c\} \,|\, \{b\} \,|\, \{f\}. \tag{122}$$

All other breakdown probabilities may be chosen arbitrarily. After deleting player f, the two top partitions become:

$$\{\{a, b, c\}, \{d, e\}\} \text{ and } \{\{a, b, c\}, \{d\}, \{e\}\}, \tag{123}$$

so that the same reduced subset $\{a, b, c\}$ is reached in two different contexts, which we denote $(\{a, b, c\}, \tau_1)$ and $(\{a, b, c\}, \tau_2)$.

To define the reduced characteristic function, we want to preserve, at each reduced context, the total value assigned originally to the players who remain. At $(\{a,b,c\},\tau_1)$*, the lift of* $\{a,b,c\}$ *to the original graph is* $\{a,b,c\}$ *itself. The deleted player* f *has no claim on this value and so we set:*

$$v^{-f}(\{a,b,c\},\tau_1) = v(\{a,b,c\}) = 0. \tag{124}$$

At $(\{a,b,c\},\tau_2)$*, the lift is* $\{a,b,c,f\}$*, and so we subtract the CCV payoff to* f *in the original game. At* $\{a,b,c,f\}$*, the local loss is* 1*, which is divided equally. We therefore set:*

$$v^{-f}(\{a,b,c\},\tau_2) = v(\{a,b,c,f\}) - \frac{1}{4} = \frac{3}{4}. \tag{125}$$

The respective local breakdown probabilities are:

$$p^{-f}_{\{a,b,c\},\tau_1}(\{a\},\{b,c\}) = 1 \text{ and } p^{-f}_{\{a,b,c\},\tau_2}(\{a,c\},\{b\}) = 1. \tag{126}$$

We see that both the reduced characteristic function and the breakdown probabilities at the two contexts differ. One might be tempted to try to define a reduced game by averaging across contexts, but then we would lose the information about which continuation game follows each partial run and the recursive structure of our solution concept would be lost.

We check consistency of the CCV in this example. In the original game, the expected local loss at the top is $-1/2$, and so each player receives $-1/12$. The expected continuation payoff vector is:

$$\frac{1}{2}\times(-\frac{1}{3},\frac{1}{6},\frac{1}{6},0,0,0)+\frac{1}{2}\times(\frac{1}{4},\frac{1}{4},\frac{1}{4},0,0,\frac{1}{4}) = (-\frac{1}{24},\frac{5}{24},\frac{5}{24},0,0,\frac{1}{8}), \tag{127}$$

so that:

$$\phi(N) = (-\frac{1}{8},\frac{1}{8},\frac{1}{8},-\frac{1}{12},-\frac{1}{12},\frac{1}{24}). \tag{128}$$

The value at the top of the reduced game is:

$$v^{-f}(N\backslash\{f\},\emptyset) = v(N) - \phi_f(N) = -\frac{1}{24}. \tag{129}$$

The expected continuation payoff vector is given by the first five entries in the vector on the right-hand side of Equation 127, which sum to 3/8, so that the reduced top expected local loss is $-1/24 - 3/8 = -5/12$. Dividing this equally among the first five players gives $-1/12$ to each, so that, adding back the continuation payoff vector, we obtain:

$$\phi^{-f}(N\backslash\{f\},\emptyset) = (-\frac{1}{8},\frac{1}{8},\frac{1}{8},-\frac{1}{12},-\frac{1}{12}), \tag{130}$$

which agrees with Equation 128.

We now lay out our general approach, using the extended CCV framework from Section 9.1. The reduction proceeds in three steps. We project each original run onto the surviving player set, keep the resulting partial run as the context of each reduced subset, and then condition on that context to define its value and its breakdown probabilities.

Definition 13. *A* ***contextual subset-refinement game*** $\mathcal{G} = (N, v, G, p)$ *is a game where, for each subset* T *of* N*, the value* $v(T,\tau)$ *and breakdown probability measure* $p_{T,\tau}$ *can vary across the partial runs* r_T*.*

Fix a player i to be deleted, write $N^{-i} = N\backslash\{i\}$, and let q be the probability measure on runs induced by the contextual breakdown probabilities. Given a partition π, let:

$$\pi^{-i} = \big\{S\backslash\{i\} \,:\, S\in\pi \text{ and } S\backslash\{i\}\neq\emptyset\big\}. \tag{131}$$

Project each run r by replacing every partition π with π^{-i}. If π^{-i} has only one cell, suppress that step and continue until the next nontrivial projected partition is reached. The resulting reduced run, denoted by r^{-i}, contains only nontrivial partitions. The family of reduced partial runs generated in this way defines a subset-refinement graph $\mathcal{G}^{-i}$ on N^{-i}.

Next, fix a context (T,τ) in $\mathcal{G}^{-i}$, and let:

$$H_{T,\tau} = \{r \,:\, (T,\tau) \text{ is reached under } r^{-i}\}. \tag{132}$$

For each $r\in H_{T,\tau}$, let $C^r(T,\tau)$ denote the first original context on the path in r that projects to the reduced context (T,τ). (If several consecutive original contexts project to (T,τ), set $C^r(T,\tau)$ to be the first of these contexts.) Letting the subset in $C^r(T,\tau)$ be S^r, we have $S^r\backslash\{i\} = T$.

We now define the reduced contextual characteristic function given a solution concept ψ. For $q(H_{T,\tau}) > 0$, set:

$$v^{-i,\psi}(T,\tau) = \mathbb{E}_q\big[\sum_{j\in T} \psi_j(C^r(T,\tau)) \;\big|\; H_{T,\tau}\big]. \tag{133}$$

At contexts reached with probability 0, define $v^{-i,\psi}(T,\tau)$ arbitrarily. Next, to define the reduced contextual breakdown probabilities, set, for $\pi \in \Pi(T)$:

$$D_{T,\tau}(\pi) = \big\{r \in H_{T,\tau} \,:\, \pi \text{ is immediately descendant from } (T,\tau) \text{ under } r^{-i}\big\}. \tag{134}$$

For $q(H_{T,\tau}) > 0$, set:

$$p^{-i}_{T,\tau}(\pi) = \frac{q\big(D_{T,\tau}(\pi)\big)}{q(H_{T,\tau})}. \tag{135}$$

At contexts reached with probability 0, define $p^{-i}_{T,\tau}$ to be an arbitrary probability measure on $\Pi(T)$.

Definition 14. *The **reduced contextual subset-refinement game relative to** i **and** ψ is:*

$$\mathcal{G}^{-i,\psi} = (N^{-i}, v^{-i,\psi}, G^{-i}, p^{-i}). \tag{136}$$

When ψ is the CCV, we write $\mathcal{G}^{-i,\psi} = \mathcal{G}^{-i}$. We can now state the key result that the reduced game preserves the CCV at every reached context.

Proposition 9. *Let (T,τ) be a context of $\mathcal{G}^{-i}$ with $q(H_{T,\tau}) > 0$. Then, for every j in T:*

$$\phi^{-i}_j(T,\tau) = \mathbb{E}_q\big[\phi_j(C^r(T,\tau)) \;\big|\; H_{T,\tau}\big]. \tag{137}$$

Definition 15. *A solution concept ψ is **consistent** if for any player i and all players $j \neq i$, we have:*

$$\psi^{-i,\psi}_j(N\backslash\{i\}) = \psi_j(N). \tag{138}$$

In words, a solution concept is consistent if, after deleting a player i, all remaining players obtain the same payoff as in the original game. We are now ready to give our analog to the consistency properties of the Shapley value established by Hart and Mas-Colell (14, 1989, Theorem B).

Theorem 6. *The CCV is consistent.*

Proof. Fix a contextual subset-refinement game $\mathcal{G}$ and a player i to be deleted. At the top reduced context, $H_{N^{-i},\emptyset}$ is the set of all runs and, for every run r, we have $C^r(N^{-i},\emptyset) = (N,\emptyset)$. Proposition 9 then gives, for every $j \neq i$:

$$\phi^{-i}_j(N^{-i},\emptyset) = \mathbb{E}_q\big[\phi_j(C^r(N^{-i},\emptyset)) \;\big|\; H_{N^{-i},\emptyset}\big] = \phi_j(N,\emptyset) = \phi_j(N), \tag{139}$$

as required. □

Theorem 7. *The CCV is the unique efficient and consistent solution concept for the family of contextual subset-refinement games that divide the gain equally in two-player games.*

Proof. The CCV is efficient, divides the gain equally in two-player games, and, by Theorem 6, is consistent. It remains to prove uniqueness. Let ψ be any solution concept with these properties. We prove that $\psi = \phi$ by induction on the number of players. The claim holds for one-player games by Efficiency and also for two-player games, since equal division uniquely determines the solution there. Suppose it holds for all games with fewer than n players, and consider an n-player game $\mathcal{G}$. Fix i in N. By consistency and the induction hypothesis applied to the two reduced games, for all $j \neq i$:

$$\psi_j(N) = \psi^{-i,\psi}_j(N\backslash\{i\}) = \phi^{-i,\psi}_j(N\backslash\{i\}), \tag{140}$$

$$\phi_j(N) = \phi^{-i,\phi}_j(N\backslash\{i\}). \tag{141}$$

Modulo contexts reached with zero probability, the reduced games $\mathcal{G}^{-i,\psi}$ and $\mathcal{G}^{-i,\phi}$ have the same contextual breakdown probabilities and the same characteristic function at every context below the top. This follows from the fact that every original context entering these reduced values contains fewer than n players, so $\psi = \phi$ there by the induction hypothesis. From this, we see that the two CCV continuation payoffs at the top are equal. The two reduced games differ only in their top values, where Efficiency gives:

$$v^{-i,\psi}(N\backslash\{i\},\emptyset) = v(N,\emptyset) - \psi_i(N), \tag{142}$$

$$v^{-i,\phi}(N\backslash\{i\},\emptyset) = v(N,\emptyset) - \phi_i(N). \tag{143}$$

Since the CCV divides the top expected local loss equally, it follows that, for any $j \neq i$:

$$\phi_j^{-i,\psi}(N\backslash\{i\}) - \phi_j^{-i,\phi}(N\backslash\{i\}) = -\frac{\psi_i(N) - \phi_i(N)}{n-1}. \tag{144}$$

But, from Equations 140 and 141, we have:

$$\phi_j^{-i,\psi}(N\backslash\{i\}) - \phi_j^{-i,\phi}(N\backslash\{i\}) = \psi_j(N) - \phi_j(N), \tag{145}$$

so that:

$$-\frac{\psi_i(N) - \phi_i(N)}{n-1} = \psi_j(N) - \phi_j(N). \tag{146}$$

Equation 146 together with the analog with i and j interchanged yields:

$$\frac{\psi_i(N) - \phi_i(N)}{(n-1)^2} = \psi_i(N) - \phi_i(N), \tag{147}$$

from which, using $n \geq 3$, we conclude that $\psi_i(N) = \phi_i(N)$ for all i, as required. □

Hart and Mas-Colell (14, 1989) also conduct a potential-theoretic analysis of the Shapley value. Given a characteristic function v, they define a potential $P : 2^N \to \mathbb{R}$ by:

$$\sum_{i \in S} \big[P(S) - P(S\backslash\{i\})\big] = v(S), \tag{148}$$

with $P(\emptyset) = 0$. The discrete gradient of this potential is the Shapley value (Hart and Mas-Colell (14, 1989, Theorem A)). For comparison, we return to our noncontextual case. In general, the CCV is not a gradient field and so does not admit a potential on the subset lattice. However, the special case that the CCV is curl-free, so that the CCV is a gradient field, turns out to provide an equivalence result, different from the approach in Section 8, between the CCV and the Shapley value.

Definition 16. *Fix a subset-refinement game $\mathcal{G} = (N, v, G, p)$. A solution concept ψ is **curl-free** if the circulation around every elementary square of the subset lattice is 0, that is, for every nonempty subset S, $|S| \geq 2$, and distinct i, j in S:*

$$\psi_i(S) + \psi_j(S\backslash\{i\}) - \psi_i(S\backslash\{j\}) - \psi_j(S) = 0. \tag{149}$$

Proposition 10. *Fix a noncontextual subset-refinement game. The CCV coincides with the Shapley value at every nonempty subset if and only if it is curl-free on the subset lattice.*

10.3 Monotonicity

Young (58, 1985) showed that the Shapley value can be axiomatized by replacing the Dummy and Additivity axioms by a Monotonicity axiom: if across two games (N, v) and (N, w), player i's marginal contribution to every subset S is weakly greater in v than w, then player i's payoff is weakly greater in v than w. We adapt Young's Monotonicity axiom to establish an analogous result in our framework. Fix two subset-refinement games $\mathcal{G}^v = (N, v, G, p)$ and $\mathcal{G}^w = (N, w, G, p)$, with respective solutions ψ^v and ψ^w.

Axiom 5. ***(Strong Monotonicity)*** *Suppose that, for all subsets U with $i \in U$:*

$$\lambda_N(U)\, m^v(U) \geq \lambda_N(U)\, m^w(U), \tag{150}$$

then $\psi_i^v \geq \psi_i^w$.

The primary adaptation of Young (58, 1985) is that we require monotonicity of the solution with respect to CCV dividends, not marginal contributions.[10] Conceptually, the idea of the axiom is that player i is at least as important in reaching multi-way agreements in game v as in game w and therefore receives at least as much in the first vs. the second game. With this axiom, we get a characterization of the CCV.

Theorem 8. *The CCV is the unique solution concept for subset-refinement games that satisfies Efficiency, Symmetry, and Strong Monotonicity.*

[10] We follow Young in referring to this condition as Strong Monotonicity, to be distinguished from Aggregate Monotonicity (Megiddo 31, 1974; Kalai and Smorodinsky 16, 1975; Balinski and Young 4, 1982), which says that if $v(N) \geq w(N)$ and $v(S) = w(S)$ for all $S \subsetneq N$, then $\psi_i^v \geq \psi_i^w$. Note that Strong Monotonicity is certainly non-vacuous if, for all subsets U with $i \in U$ and all $\pi \in \Pi(U)$, we have $\lambda_N(U)\, l_\pi^v(U) \geq \lambda_N(U)\, l_\pi^w(U)$.

The argument in Young (58, 1985) decomposes a game into the unanimity basis and proceeds via induction on unanimity-basis coefficients. Our method is analogous, but proceeds via induction on the number of nonzero CCV dividends, $\lambda_N(S)\, m(S)$.

10.4 A Non-Cooperative Route

We formulate a non-cooperative game whose backward-induction solution yields the CCV expected payoffs to the players. Our starting point is the Hart and Mas-Colell (15, 1996) implementation of the Shapley value, where a player is chosen uniformly at random to propose an allocation to the other active players, who then sequentially accept or reject it. If unanimously accepted, the proposed allocation is realized. Otherwise, the proposer leaves the game with some probability and the process repeats.

We consider a finite-horizon construction. For every nonempty subset S, define a game recursively. If $S = \{i\}$, the game terminates with payoff $v(\{i\})$ to player i. For $|S| \geq 2$, Nature chooses a proposer k in S uniformly at random. The proposer chooses an allocation $x \in \mathbb{R}^S$ satisfying $\sum_{i \in S} x_i \leq v(S)$. The other players in S respond in a fixed and observable order. If unanimously accepted, the allocation x is implemented and the game at S terminates. At the first rejection (if there is one), the proposal is discarded and Nature chooses a partition $\pi \in \Pi(S)$ with probability $p_S(\pi)$. The cells $T \in \pi$ are then considered in some fixed order and, for each cell T, the associated game is played. The resulting payoff vectors from the cells are combined to form the overall continuation payoff vector for S.

The tie-breaking assumption is that a responder who is indifferent between acceptance and rejection accepts and a proposer who is indifferent between making a proposal that would be accepted and one that would be rejected chooses to make a proposal that would be accepted. We also assume that all expected local losses $m(S)$ are non-negative, for which a sufficient condition is superadditivity. The proposal sets are continua but, since every rejection replaces a subset by a strict partition, the game has finite depth and we may apply backward induction.

Proposition 11. *There is a unique backward-induction payoff to the game. Agreement occurs immediately at every subset. Before the proposer is selected for a subset S, the expected payoff to each player $i \in S$ is the CCV payoff $\phi_i(S)$. In particular, before the initial proposer is selected, the expected payoff vector is $\phi(N)$.*

The comparison with Hart and Mas-Colell (15, 1996) is most direct in their special case when rejection removes the proposer. If standalone values are zero-normalized, then, if player k is the proposer, the other players must collectively receive their continuation value $v(S\backslash\{k\})$ and so the proposer gets $v(S) - v(S\backslash\{k\})$, which is its marginal contribution. In our game, the partition selected following rejection is independent of the identity of the proposer. Therefore, every proposer sees the same expected gain $m(S)$ from agreement relative to breakdown. The proposer captures this gain conditional on being selected, so that, given random proposer selection, the expected gain to each member of S is $m(S)/|S|$. This is our non-cooperative counterpart to the collective vs. individualistic treatment of loss underlying the CCV.

10.5 Regression

A standard result in game theory and machine learning is that the Shapley value of a cooperative game (N, v) is the unique vector x solving the weighted least-squares problem (Charnes et al. 7, 1988; Lundberg and Lee 22, 2017; Fernández-Loría, Provost, and Han 11, 2022; Rozemberczki et al. 43, 2022):

$$\min_{(x_i)_{i \in N}} \sum_{\emptyset \neq S \subsetneq N} w(S) \left(v(S) - \sum_{i \in S} x_i\right)^2, \tag{151}$$

$$\text{subject to } \sum_{i \in N} x_i = v(N), \tag{152}$$

$$\text{where } w(S) = \frac{(|S|-1)!(|N\backslash S|-1)!}{(|N|-1)!}. \tag{153}$$

To obtain an analogous representation of the CCV, we construct an auxiliary cooperative game (N, v') by setting:

$$v'(T) = \sum_{\emptyset \neq U \subseteq T} \lambda_N(U)\, m(U). \tag{154}$$

Here, as before, the Harsanyi dividend assigned to U in the auxiliary game is the expected local loss $m(U)$ weighted by the hitting probability $\lambda_N(U)$.

Proposition 12. *The CCV is the unique solution of the weighted least-squares minimization problem for v'.*

Proof. By Möbius inversion, the Harsanyi dividend of v' at U is $\lambda_N(U)\, m(U)$. Therefore, the Shapley value of the auxiliary game assigns player i:

$$\sum_{i \in U \subseteq N} \frac{\lambda_N(U)\, m(U)}{|U|}, \tag{155}$$

which is equal to the CCV by Proposition 2. The result follows from the least-squares characterization of the Shapley value. □

11 Discussion

We comment on some conceptual aspects of our analysis and point to some future directions.

a. Exogenous vs. Endogenous Probabilities We have augmented the traditional description of a cooperative game by adding a graph structure and breakdown probabilities. For these probabilities, we considered both exogenous and endogenous formulations. Initially, the subset-refinement game and the CCV itself are defined using arbitrary, exogenous probability measures. We continued this approach through our analysis of the CCV polytope in Section 4, the axiomatization in Section 5, and the use of Möbius and 1BB weights in Section 8.

Taking the breakdown probabilities as exogenously given is a natural modeling choice. Just as the characteristic function is treated as an exogenous primitive of the game, so can the breakdown probabilities. They may reflect factors external to the game as described, such as existing relationships among subsets of players or historical data on the likelihoods of different coalitions forming.

We also considered cases where the breakdown probabilities are endogenous to the characteristic function. In Section 6, we introduced loss-harmonic probabilities, where partitions associated with larger losses are less likely to arise. Likewise, in Examples 9–11 in Section 7, we imposed endogenous behavioral constraints—specifically, that probabilities are non-increasing in losses and select zero-loss exits uniformly. Endogeneity is often natural. Consider again Example 2, where players a and b achieve the maximum of 1 without c. It is reasonable, but not inevitable, that player c is left out in the event of non-agreement.

b. Indeterminism vs. Determinism The choice between exogenous and endogenous probabilities is closely tied to whether the CCV yields a single deterministic point or an indeterminate set of expected payoff vectors. When the breakdown probabilities are permitted to vary arbitrarily, the CCV generates a polytope of solutions, as established in Theorem 1. This indeterminacy is evident in the examples of Section 7, where placing interpretable constraints on the probabilities narrows the CCV to a circumscribed range of outcomes but does not force a single point. We view this indeterminacy not as a defect, but as a deliberate modeling choice that allows external factors to matter, as already mentioned. A report (Kuhn and Tucker, 19, 1958) on a 1955 meeting on game theory describes von Neumann's view that:

> [T]he enormous variety of solutions which may obtain for n-person games was not surprising in view of the correspondingly enormous variety of observed stable social structures; many differing conventions can endure, existing today for no better reason than that they were here yesterday.

While von Neumann's specific focus was on the value of the game (von Neumann and Morgenstern 56, 1944), it is clear that he saw cooperative theory as a consciously indeterminate analysis. We view the CCV similarly, but we are also interested in the determinate case.

The CCV collapses to a single point both when the breakdown probabilities are fixed and when they are endogenous but fully specified – as in the case of the loss-harmonic rule. Notably, the CCV also collapses to a single point under unit hitting probabilities, in which case it then coincides with the Shapley value (Theorem 4).

c. Gains vs. Marginal Contributions We next return to Example 2 and take a closer look at how the CCV and the Shapley value operate there. The CCV allocates losses (or gains) associated with breakdown, while the Shapley value is usually described as allocating marginal contributions. In fact, both solution concepts can be understood as allocating gains from cooperation, where the difference arises in how these gains are divided. If player i exits from a subset S, the associated loss is:

$$v(S) - v(S\backslash\{i\}) - v(\{i\}). \tag{156}$$

In addition to a share of this loss, the CCV gives player i its standalone value $v(\{i\})$. If, instead, the entire loss is allocated to i, its total payoff would be:

$$v(\{i\}) + \big[v(S) - v(S\backslash\{i\}) - v(\{i\})\big] = v(S) - v(S\backslash\{i\}), \tag{157}$$

which is exactly player i's marginal contribution.

This suggests a useful re-framing of the usual Shapley procedure. Begin by giving the players their standalone values. Then let the players join in a random order. When player i joins an existing subset T, the gain from the resulting cooperation is:

$$v(T \cup \{i\}) - v(T) - v(\{i\}). \tag{158}$$

The Shapley value effectively assigns this entire gain to the joining player. Adding player i's standalone value gives:

$$v(\{i\}) + \big[v(T \cup \{i\}) - v(T) - v(\{i\})\big] = v(T \cup \{i\}) - v(T), \tag{159}$$

which is the usual marginal contribution. Now view the CCV in the same way. Players again begin with their standalone values. The difference is that the gain from cooperation of Equation 158 is shared among the players jointly involved in the agreement rather than assigned entirely to the joining player.

In sum, the distinction between the Shapley value and the CCV need not be framed as one between marginal contributions and gains. Both can be viewed as allocating the gains from cooperation. The Shapley value assigns each gain entirely to the player whose arrival creates it. The CCV shares the gain among the players whose agreement creates it.

d. Ex Post vs. Ex Ante Agreement We saw in Examples 12 and 13 that externalities naturally give rise to nonsuperadditive characteristic functions and thereby negative expected local losses. This raises an interesting issue. If $m(S) < 0$ for some subset S, then, when S is reached, the players may prefer not to agree on dividing the negative number $m(S)$ and, instead, let S actually break down. Indeed, writing the CCV recursion as:

$$\phi_i(S) - \sum_{\pi \in \Pi(S)} p_S(\pi)\, \phi_i(\pi_i) = \frac{m(S)}{|S|}, \tag{160}$$

we see that, if $m(S) < 0$, every member of S strictly prefers the breakdown lottery to agreement. We might therefore posit an ex post agreement rule saying that once a subset is reached, agreement occurs only if the gain relative to continuation is nonnegative, and actual breakdown occurs otherwise. Continuation values are then computed recursively under this rule.

There is an interesting ex ante-ex post tension. The ex post rule may make some players worse off overall than under plain CCV. Ex ante, some players may therefore prefer to commit to agreement at all subsets, including those with negative $m(S)$. The equal-division rule of the CCV is predicated on the idea that all relevant players are essential for an agreement. If moving to the ex post rule is not unanimously agreed on, then, perhaps, this may not happen. Another interesting aspect is that adoption of the ex post agreement rule will, in general, violate our Additivity axiom.

When breakdown weights are signed, there is an additional reason to look for an ex post extension. Example 13 shows that the upper-Möbius calculation can produce extremely large individual payoffs even when the characteristic function is bounded. This phenomenon can be understood through the proliferation of partial runs in the presence of externalities. The full contextual subset-refinement graph has 16 partial runs with three players, 121 with four players, and 1,151 with five players. With 101 players, as in Example 13, the number of partial runs is approximately 10^{187}. Each partial run leads to a distinct context at which gains and losses can enter the calculation. Under the upper Möbius distribution, the resulting contributions can be large and of opposite signs. Although these quantities must cancel in the aggregate to satisfy Efficiency, they need not cancel player by player and we saw they may blow up.

We hypothesize that blowup can be controlled via a suitable extension of the ex post agreement rule to games with signed breakdown weights. This extension is not immediate, because a signed expected local loss need not describe an operationalizable lottery.

e. Future Directions In Section 8, we studied signed breakdown weights in the form of the upper and lower Möbius distributions. The latter, in particular, pointed towards an extended symmetry principle. These weights can be interpreted as a mathematical reconciliation of symmetric but mutually inconsistent, player-specific perspectives on how the grand coalition will break down. A careful and more general treatment of this idea is left open.

Turning to large-N games and asymptotics, we conducted a direct analysis for the Glove Game of Example 8. More generally, since the CCV can be viewed as an expected accumulated reward over a refinement process, asymptotic methods for Markov-reward processes may offer tools for studying large-N behavior in other games of applied interest.

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Appendix A: Proofs for Section 3

Proof of Proposition 1. We show the CCV is efficient via induction. The base case is immediate. Suppose, inductively, that the result is true for all sets of size less than or equal to $n-1$, and consider a set S with $|S| = n$. Summing over the values $\phi_i(S)$ in Equation 20 yields:

$$\sum_{i\in S}\phi_i(S) = \sum_{i\in S}\sum_{\pi\in\Pi(S)} p_S(\pi)\big[\frac{v(S)-\sum_{T\in\pi}v(T)}{|S|}+\phi_i(\pi_i)\big] \tag{A.1}$$

$$= \sum_{\pi\in\Pi(S)}\sum_{i\in S} p_S(\pi)\big[\frac{v(S)-\sum_{T\in\pi}v(T)}{|S|}+\phi_i(\pi_i)\big] \tag{A.2}$$

$$= \sum_{\pi\in\Pi(S)} p_S(\pi)\big[v(S)-\sum_{T\in\pi}v(T)+\sum_{i\in S}\phi_i(\pi_i)\big] \tag{A.3}$$

$$= v(S) - \sum_{\pi\in\Pi(S)} p_S(\pi)\sum_{T\in\pi}\big[v(T)-\sum_{i\in T}\phi_i(T)\big] \tag{A.4}$$

$$= v(S). \tag{A.5}$$

The key step arises in Equation A.4. Since each set T is a member of the nontrivial partition π of S, we know that $|T| \le n-1$. By the induction hypothesis, $\sum_{i\in T}\phi_i(T) = v(T)$, which we substitute in to get the result. □

Proof of Proposition 2. Using $\lambda_S(S) = 1$ and substituting in the recursion of Equation 25, we get:

$$\phi_i^*(S) = \lambda_S(S)\,\frac{m(S)}{|S|} + \sum_{i\in T\subsetneq S}\lambda_S(T)\,\frac{m(T)}{|T|} \tag{A.6}$$

$$= \frac{m(S)}{|S|} + \sum_{i\in T\subsetneq S}\big[\sum_{\pi\in\Pi(S)} p_S(\pi)\,\lambda_{\pi_i}(T)\big]\frac{m(T)}{|T|}. \tag{A.7}$$

For a fixed T with $i \in T$, if $T \nsubseteq \pi_i$, then, by our convention, we have $\lambda_{\pi_i}(T) = 0$. With this, we can interchange the order of summation to write:

$$\phi_i^*(S) = \frac{m(S)}{|S|} + \sum_{\pi \in \Pi(S)} p_S(\pi) \sum_{i \in T \subseteq \pi_i} \lambda_{\pi_i}(T) \frac{m(T)}{|T|}. \tag{A.8}$$

The inner sum is exactly $\phi^*(\pi_i)$, so we obtain the same recursion as Equation 23, and the initial conditions agree. We conclude that $\phi_i^*(S) = \phi_i(S)$, as required. □

Proof of Proposition 3. Fix a nonsingleton subset T and a partition $\pi = \{T_1, \ldots, T_k\} \in \Pi(T)$. Since the cells of π are pairwise disjoint and have union T, repeated application of superadditivity gives

$$v(T) \geq \sum_{j=1}^{k} v(T_j). \tag{A.9}$$

It follows that $l_\pi(T) \geq 0$, from which, for every nonsingleton T:

$$m(T) = \sum_{\pi \in \Pi(T)} p_T(\pi)\, l_\pi(T) \geq 0. \tag{A.10}$$

Now use Equation 30, $\lambda_S(T) \geq 0$, and $\lambda_S(\{i\}) = 1$ to conclude that $\phi_i^*(S) \geq \lambda_S(\{i\}) \frac{v(\{i\})}{1} = v(\{i\})$, so that, by Proposition 2, we obtain $\phi_i(S) \geq v(\{i\})$, as required. □

Proof of Proposition 4. First consider $T = \{i\}$. We have:

$$m'(\{i\}) = v'(\{i\}) = \mu \cdot v(\{i\}) + \nu_i = \mu \cdot m(\{i\}) + \nu_i. \tag{A.11}$$

Now consider T with $|T| \geq 2$. Then:

$$m'(T) = \sum_{\pi \in \Pi(T)} p_T(\pi) \big[v'(T) - \sum_{U \in \pi} v'(U)\big] \tag{A.12}$$

$$= \sum_{\pi \in \Pi(T)} p_T(\pi) \big[\mu \cdot v(T) + \sum_{i \in T} \nu_i - \sum_{U \in \pi} (\mu \cdot v(U) + \sum_{j \in U} \nu_j)\big]. \tag{A.13}$$

Since the sets $U \in \pi$ form a partition of T, we have:

$$\sum_{U \in \pi} \sum_{j \in U} \nu_j = \sum_{i \in T} \nu_i, \tag{A.14}$$

so that all additive constants cancel in Equation A.13 and we obtain:

$$m'(T) = \sum_{\pi \in \Pi(T)} p_T(\pi)\, \mu\big[v(T) - \sum_{U \in \pi} v(U)\big] = \mu \cdot m(T). \tag{A.15}$$

Next substitute Equations A.11 and A.15 into Equation 30 for the transformed game. For i in S:

$$(\phi_i^*)'(S) = \lambda_S(\{i\}) \frac{m'(\{i\})}{1} + \sum_{\substack{i \in T \subseteq S \\ |T| \geq 2}} \lambda_S(T) \frac{m'(T)}{|T|} = \mu m(\{i\}) + \nu_i + \sum_{\substack{i \in T \subseteq S \\ |T| \geq 2}} \lambda_S(T) \frac{\mu m(T)}{|T|} \tag{A.16}$$

$$= \mu \sum_{i \in T \subseteq S} \lambda_S(T) \frac{m(T)}{|T|} + \nu_i = \mu\, \phi_i^*(S) + \nu_i, \tag{A.17}$$

where we separate the singleton term in the summation and use $\lambda_S(\{i\}) = 1$. The CCV formula in Proposition 2 then establishes the result. □

Appendix B: Proofs for Section 4

We first establish a lemma showing that the hitting probabilities obtained by averaging several breakdown probability families are generated by a valid family of local breakdown probabilities. This bridge allows us to invoke the second part of Axiom 4 (Additivity) and conclude that the CCV solution set is convex. We then identify the deterministic payoff vectors with the pure-strategy payoffs of an associated extensive-form game

and apply Kuhn's theorem (Kuhn 17, 1950, 18, 1953) to conclude that the solution set is the convex hull of these vectors. The same argument immediately yields Corollary B.1 for occurrence-dependent breakdown probabilities.

Fix $p^1, \ldots, p^K \in \mathcal{B}$ and $\alpha_1, \ldots, \alpha_K \geq 0$ with $\sum_{k=1}^K \alpha_k = 1$. Write $\lambda_N^k(S)$ for the hitting probability of S under p^k, and define:

$$\overline{\lambda}_N(S) = \sum_{k=1}^K \alpha_k \lambda_N^k(S), \tag{B.1}$$

for every nonempty S. For every nonsingleton S, define:

$$\overline{p}_S(\pi) = \frac{\sum_{k=1}^K \alpha_k \lambda_N^k(S) p_S^k(\pi)}{\overline{\lambda}_N(S)}, \tag{B.2}$$

whenever $\overline{\lambda}_N(S) > 0$, and choose $\overline{p}_S$ arbitrarily from $\Delta(\Pi(S))$ otherwise.

Lemma B.1. *We have $\overline{p} \in \mathcal{B}$ and the hitting probabilities generated by $\overline{p}$ satisfy:*

$$\lambda_N^{\overline{p}}(S) = \overline{\lambda}_N(S), \tag{B.3}$$

for every nonempty S.

Proof. Suppose first that $\overline{\lambda}_N(S) > 0$. Each term in the numerator of Equation B.2 is non-negative, and:

$$\sum_{\pi \in \Pi(S)} \overline{p}_S(\pi) = \frac{\sum_{k=1}^K \alpha_k \lambda_N^k(S) \sum_{\pi \in \Pi(S)} p_S^k(\pi)}{\overline{\lambda}_N(S)} = \frac{\sum_{k=1}^K \alpha_k \lambda_N^k(S)}{\overline{\lambda}_N(S)} = 1. \tag{B.4}$$

When $\overline{\lambda}_N(S) = 0$, we have chosen $\overline{p}_S$ to be an arbitrary probability measure. Therefore, $\overline{p} \in \mathcal{B}$.

For any $q \in \mathcal{B}$ and any nonempty strict subset T of N, the hitting probability of T satisfies:

$$\lambda_N^q(T) = \sum_{S \supsetneq T} \sum_{\{\pi \in \Pi(S) \,:\, T \in \pi\}} \lambda_N^q(S) q_S(\pi). \tag{B.5}$$

The definition of $\overline{p}$ gives, for every nonsingleton S and $\pi \in \Pi(S)$:

$$\overline{\lambda}_N(S) \overline{p}_S(\pi) = \sum_{k=1}^K \alpha_k \lambda_N^k(S) p_S^k(\pi). \tag{B.6}$$

This is immediate when $\overline{\lambda}_N(S) > 0$. If $\overline{\lambda}_N(S) = 0$, then non-negativity implies $\alpha_k \lambda_N^k(S) = 0$ for every k, and so both sides of Equation B.6 are 0.

We now prove equality of the hitting probabilities by induction. At the top of the graph:

$$\lambda_N^{\overline{p}}(N) = 1 = \sum_{k=1}^K \alpha_k \lambda_N^k(N) = \overline{\lambda}_N(N). \tag{B.7}$$

Fix a nonempty strict subset T of N and suppose, inductively, that:

$$\lambda_N^{\overline{p}}(S) = \overline{\lambda}_N(S), \tag{B.8}$$

for every strict superset S of T. Using Equation B.5, the induction hypothesis, and Equation B.6, we get:

$$\lambda_N^{\overline{p}}(T) = \sum_{S \supsetneq T} \sum_{\{\pi \in \Pi(S) \,:\, T \in \pi\}} \lambda_N^{\overline{p}}(S) \overline{p}_S(\pi) = \sum_{S \supsetneq T} \sum_{\{\pi \in \Pi(S) \,:\, T \in \pi\}} \overline{\lambda}_N(S) \overline{p}_S(\pi) \tag{B.9}$$

$$= \sum_{S \supsetneq T} \sum_{\{\pi \in \Pi(S) \,:\, T \in \pi\}} \sum_{k=1}^K \alpha_k \lambda_N^k(S) p_S^k(\pi). \tag{B.10}$$

Interchanging the order of summation gives:

$$\lambda_N^{\overline{p}}(T) = \sum_{k=1}^K \alpha_k \sum_{S \supsetneq T} \sum_{\{\pi \in \Pi(S) \,:\, T \in \pi\}} \lambda_N^k(S) p_S^k(\pi). \tag{B.11}$$

Applying Equation B.5 to each p^k, we get:

$$\lambda_N^{\overline{p}}(T) = \sum_{k=1}^{K} \alpha_k \lambda_N^k(T) = \overline{\lambda}_N(T), \tag{B.12}$$

which completes the induction. □

Proof of Theorem 1. We first show that $\mathcal{W}$ is convex. Fix $p^1, \ldots, p^K \in \mathcal{B}$ and $\alpha_1, \ldots, \alpha_K \geq 0$ with $\sum_{k=1}^{K} \alpha_k = 1$, and let $\overline{p}$ be the family of breakdown probability measures constructed in Lemma B.1. We know that $\overline{p} \in \mathcal{B}$ and its hitting probabilities satisfy Equation B.3 for every nonempty S. The second part of Axiom 4 (Additivity) for the CCV (proved in Appendix C) therefore gives:

$$w(\overline{p}) = \sum_{k=1}^{K} \alpha_k w(p^k), \tag{B.13}$$

establishing that $\mathcal{W}$ is convex. Since every deterministic breakdown family belongs to $\mathcal{B}$, we have $\mathcal{V} \subseteq \mathcal{W}$, and therefore $\text{conv}(\mathcal{V}) \subseteq \mathcal{W}$.

We now cast the subset-refinement game as a one-player finite extensive-form game. Associate a separate decision node with every occurrence of every nonsingleton subset S, and allow the breakdown probability measure at S to depend on the occurrence. Let $\mathcal{W}^e$ denote the set of CCV expected-payoff vectors generated by behavior strategies in this enhanced game, and let $\mathcal{V}^e$ denote the set of payoff vectors generated by its pure strategies. The enhanced game has perfect information and hence perfect recall. Therefore, Kuhn's theorem (Kuhn 17, 1950, 18, 1953) implies $W^e = \text{conv}(\mathcal{V}^e)$.

We next show that $\mathcal{V}^e = \mathcal{V}$. First, take an original deterministic family and use its selected partition at every occurrence of the corresponding subset in the enhanced game. This produces the same deterministic run and therefore the same CCV payoff vector. Therefore, $\mathcal{V} \subseteq \mathcal{V}^e$. Conversely, fix a pure strategy in the enhanced game. In the induced deterministic run, any given nonempty subset S is reached at most once. For every nonsingleton subset S that is reached, let an original deterministic family select the partition chosen at its unique reached occurrence. At every subset that is not reached, make an arbitrary selection. Proceeding down from the top of the graph, this original deterministic family induces exactly the same realized refinement and hence the same CCV payoff vector. Therefore, $\mathcal{V}^e \subseteq \mathcal{V}$, proving the claim.

The original game is a special case of the enhanced game, so $\mathcal{W} \subseteq \mathcal{W}^e$. Combining these results, we obtain:

$$\mathcal{W} \subseteq \mathcal{W}^e = \text{conv}(\mathcal{V}^e) = \text{conv}(\mathcal{V}) \subseteq \mathcal{W}, \tag{B.14}$$

from which:

$$\mathcal{W} = \text{conv}(\mathcal{V}). \tag{B.15}$$

Finally, N is finite and each $\Pi(S)$ is finite, so the set of deterministic families is finite. This establishes that $\mathcal{V}$ is finite and $\mathcal{W}$ is a polytope. □

Corollary B.1. *Consider the enhanced subset-refinement game in which the breakdown probability at a subset S may depend on the particular occurrence of S in the graph. Let $\mathcal{W}^e$ denote the set of CCV expected-payoff vectors generated by this enhanced game. Then $\mathcal{W}^e = \mathcal{W}$.*

We observe that the occurrence-dependent breakdown probabilities do not expand the set of CCV expected-payoff vectors.

Proof. The equality $\mathcal{W}^e = \mathcal{W}$ was established in the proof of Theorem 1. □

Appendix C: Proofs for Section 5

Proof of Necessity in Theorem 2. Efficiency is established in Proposition 1. For the remaining properties, we use the formula for the CCV in Proposition 2. For Symmetry, write:

$$\phi_i^*(N) - \phi_j^*(N) = \sum_{i \in T \subseteq N} \lambda_N(T) \frac{m(T)}{|T|} - \sum_{j \in T \subseteq N} \lambda_N(T) \frac{m(T)}{|T|} \tag{C.1}$$

$$= \sum_{\substack{i \in T \subseteq N \\ j \notin T}} \lambda_N(T) \frac{m(T)}{|T|} - \sum_{\substack{j \in T \subseteq N \\ i \notin T}} \lambda_N(T) \frac{m(T)}{|T|} \tag{C.2}$$

$$= \sum_{S \subseteq N \setminus \{i,j\}} \frac{\lambda_N(S \cup \{i\})\, m(S \cup \{i\}) - \lambda_N(S \cup \{j\})\, m(S \cup \{j\})}{|S| + 1}, \quad \text{(C.3)}$$

where, in the penultimate line, we re-index sets T containing i but not j as $T = S \cup \{i\}$, for $S \subseteq N \setminus \{i, j\}$, and similarly for j. The denominator $|S| + 1$ comes from $|T|$. By Symmetry, each term in the numerator is 0 and so $\phi_i^*(N) = \phi_j^*(N)$, as required.

Next, observe that if player i is a dummy, we have:

$$\phi_i^*(N) = \sum_{i \in T \subseteq N} \lambda_N(T)\, \frac{m(T)}{|T|} = \lambda_N(\{i\})\, m(\{i\}) = v(\{i\}), \quad \text{(C.4)}$$

since $\lambda_N(T)\, m(T) = 0$ for each nonsingleton subset T in the sum and $\lambda_N(\{i\}) = 1$.

For the first case of Additivity with constant breakdown probabilities, the hitting probabilities $\lambda_N^k(T)$ are constant over k, and we have $m(T) = \sum_k m^k(T)$. Therefore:

$$\phi_i^*(N) = \sum_{i \in T \subseteq N} \lambda_N(T)\, \frac{m(T)}{|T|} = \sum_{i \in T \subseteq N} \lambda_N(T)\, \frac{\sum_k m^k(T)}{|T|} = \sum_k \phi_i^{k,*}(N). \quad \text{(C.5)}$$

For the second case of Additivity with constant characteristic functions, the local losses $l_\pi^k(T)$ are constant over k. Therefore:

$$\sum_k \alpha^k \phi_i^{*,k}(N) = \sum_k \alpha^k \sum_{i \in T \subseteq N} \lambda_N^k(T)\, \frac{m^k(T)}{|T|} = \sum_k \alpha^k \big[v(\{i\}) + \sum_{\substack{i \in T \subseteq N \\ |T| \geq 2}} \lambda_N^k(T) \sum_{\pi \in \Pi(T)} p_T^k(\pi) \frac{l_\pi(T)}{|T|} \big] \quad \text{(C.6)}$$

$$= v(\{i\}) + \sum_{\substack{i \in T \subseteq N \\ |T| \geq 2}} \sum_{\pi \in \Pi(T)} \frac{l_\pi(T)}{|T|} \sum_k \alpha^k\, \lambda_N^k(T)\, p_T^k(\pi) \quad \text{(C.7)}$$

$$= v(\{i\}) + \sum_{\substack{i \in T \subseteq N \\ |T| \geq 2}} \sum_{\pi \in \Pi(T)} \frac{l_\pi(T)}{|T|} \lambda_N(T) p_T(\pi) \quad \text{(C.8)}$$

$$= v(\{i\}) + \sum_{\substack{i \in T \subseteq N \\ |T| \geq 2}} \lambda_N(T) \sum_{\pi \in \Pi(T)} \frac{l_\pi(T)}{|T|}\, p_T(\pi) = \sum_{i \in T \subseteq N} \lambda_N(T)\, \frac{m(T)}{|T|} = \phi_i^*(N), \quad \text{(C.9)}$$

as required. □

Proof of Lemma 1. Case (i): Fix $k \notin U$. By construction of the game, no local loss ever occurs at any subset containing k. Therefore, player k is a dummy, and by the Dummy axiom, $\psi_k(\mathcal{G}_U^r) = v_U^r(\{k\}) = 0$.

Case (ii): By part (i), all players outside U receive 0. The remaining question is whether $\lambda_N(U) = 0$ or 1. (These are the only two cases in a deterministic game; the answer depends on whether U is reached in the run or not.) If $U = \{k\}$, then $\lambda_N(U) = 1$. By Efficiency, player k must receive $l_r(U)/|U| = v(\{k\})$. Next consider a subset U with $|U| \geq 2$. If $\lambda_N(U) = 0$, then all players in U are dummy players and therefore receive $\psi_k(\mathcal{G}_U^r) = v_U^r(\{k\}) = 0$. If $\lambda_N(U) = 1$, the only value to be allocated is the realized local loss $l_r(U)$, and, by Efficiency, it must therefore be fully allocated among the player(s) in U. Consider any two distinct players $i, j \in U$ (recall that $|U| \geq 2$). For any subset S with $i, j \notin S$, we have $m(S \cup \{i\}) = m(S \cup \{j\}) = 0$, since the only subset that has a nonzero local loss is U. Therefore, the hypothesis of the Symmetry axiom holds for i, j, and all players in U must receive the same amount. By Efficiency, these payoffs must sum to $l_r(U)$, so each player in U receives $l_r(U)/|U|$. □

Proof of Lemma 2. Observe that, by construction:

$$\sum_{\emptyset \neq U \subseteq N} v_U^r(W) = \sum_{U \in \mathcal{R}(r,W)} l^r(U). \quad \text{(C.10)}$$

Iterating Equation 58 on the run starting from W yields the telescoping sum:

$$\sum_{\substack{U \in \mathcal{R}(r,W) \\ |U| \geq 2}} \Big[v(U) - \sum_{T \in \pi^r(U)} v(T) \Big] + \sum_{\{i\} \in \mathcal{R}(r,W)} v(\{i\}) = v(W), \quad \text{(C.11)}$$

Every reached subset $U \subsetneq W$ appears once with a negative sign in the local loss of its unique parent on the

run. If U is nonsingleton, it appears once with a positive sign in its own local loss; if $U = \{i\}$, it appears positively through the convention $l^r(\{i\}) = v(\{i\})$. Hence all terms cancel except the $v(W)$ at the root. □

Proof of Lemma 3. By Lemma 2 we can apply Additivity in its first form to Equation 59. For each player i:

$$\psi_i(\mathcal{G}^r) = \sum_{i \in U \subseteq N} \psi_i(\mathcal{G}^r_U) = \sum_{i \in U \subseteq N} \lambda_N(U)\, \frac{l_r(U)}{|U|} = \sum_{i \in U \in \mathcal{R}(r,N)} \frac{l_r(U)}{|U|}, \tag{C.12}$$

where the final equality follows from Lemma 1. □

Proof of Lemma 4. The game G is the formal mixture of the deterministic games G^r, weighted by the probability $q(r)$ of each run r. To see this, note first that, for every nonempty subset U,

$$\sum_r q(r) \lambda^r_N(U) = \lambda_N(U), \tag{C.13}$$

since $\lambda^r_N(U)$ is the indicator that U is reached on run r. Moreover, the event that U is reached depends only on the partition choices at strict supersets of U, while $\pi^r(U)$ is drawn independently under p_U. Therefore, conditional on U being reached, the distribution of $\pi^r(U)$ remains p_U. It follows that the mixture of the deterministic games G^r according to $q(r)$ reproduces both the hitting probabilities and the breakdown probabilities of G.

By the Additivity axiom in its second form, we get:

$$\psi_i(\mathcal{G}) = \mathbb{E}_r\big[\psi_i(\mathcal{G}^r)\big]. \tag{C.14}$$

Substituting the run-level payoffs from Lemma 3 yields:

$$\psi_i(\mathcal{G}) = \mathbb{E}_r\big[\sum_{i \in U \in \mathcal{R}(r,N)} \frac{l_r(U)}{|U|}\big]. \tag{C.15}$$

Interchanging the expectation and summation gives:

$$\psi_i(\mathcal{G}) = \sum_{i \in U \subseteq N} \frac{1}{|U|}\, \mathbb{E}_r\big[\mathbf{1}_{\{U \in \mathcal{R}(r,N)\}}\, l_r(U)\big]. \tag{C.16}$$

For a nonsingleton U, the event that U is reached depends only on the partitions selected at strict supersets of U, while $l^r(U)$ depends only on the partition selected at U. Under the product measure q, these are independent. It follows that:

$$\mathbb{E}_r\left[l^r(U) \mid U \in R(r,N)\right] = \mathbb{E}_r\left[l^r(U)\right] = m(U), \tag{C.17}$$

whenever $\lambda_N(U) > 0$. (If $\lambda_N(U) = 0$, the conditional expectation is left undefined.) For a singleton $U = \{i\}$, the same equality follows directly from:

$$l^r(\{i\}) = m(\{i\}) = v(\{i\}). \tag{C.18}$$

We now have:

$$\mathbb{E}_r\big[\mathbf{1}_{\{U \in \mathcal{R}(r,N)\}}\, l_r(U)\big] = \mathbb{E}_r\big[l_r(U) | U \in \mathcal{R}(r,N)\big]\, \mathbb{P}\big(U \in \mathcal{R}(r,N)\big) = m(U)\, \lambda_N(U), \tag{C.19}$$

and substituting in Equation C.16 yields Equation 62, as required. □

Appendix D: Proofs for Section 6

Proof of Theorem 3. For each subset S with $|S| \geq 3$, set $q_i(S) = p_S(\{\{i\}, S\backslash\{i\}\})$ for each i in S. For $S \subseteq N$ with $|S| = 2$, we adopt the convention that $q_i(S) = 1/2$ for each $i \in S$.

We first show that $m(S)/|S| = H(S)/|S| = q_i(S)\, l^i(S)$ for all i in S. For $|S| = 2$, we have $l^i(S) = l^j(S)$, so that the convention $q_i(S) = q_j(S) = 1/2$ gives $m(S) = H(S)$ and $q_i(S)\, l^i(S) = q_j(S)\, l^j(S) = H(S)/2$.

Now assume $|S| \geq 3$. If all $l^i(S) > 0$, then, by Definition 9:

$$m(S) = \sum_{j \in S} q_j(S)\, l^j(S) = \frac{H(S)}{|S|} \sum_{j \in S} \frac{1}{l^j(S)} \times l^j(S) = H(S). \tag{D.1}$$

If some $l^i(S) = 0$, it still follows that $m(S) = H(S)$ since both quantities are equal to 0. (This uses the fact that, in this case, $q_j(S) > 0$ only if $l^j(S) = 0$.) It follows from Equations 69 and 70 that, for all j in S:

$$q_j(S)\, l^j(S) = \frac{H(S)}{|S|}. \tag{D.2}$$

Putting this together, we rewrite the recursion in Equation 23 as:

$$\begin{aligned}
\phi_i(S) &= \frac{m(S)}{|S|} + \sum_{j \in S\backslash\{i\}} q_j(S)\, \phi_i(S\backslash\{j\}) + q_i(S)\, \phi_i(\{i\}) && \text{(D.3)}\\
&= \frac{H(S)}{|S|} + \sum_{j \in S\backslash\{i\}} q_j(S)\, \phi_i(S\backslash\{j\}) + q_i(S)\, \phi_i(\{i\}) && \text{(D.4)}\\
&= q_i(S)\, l^i(S) + \sum_{j \in S\backslash\{i\}} q_j(S)\, \phi_i(S\backslash\{j\}) + q_i(S) v(\{i\}) && \text{(D.5)}\\
&= q_i(S)\, \big[v(S) - v(S\backslash\{i\})\big] + \sum_{j \in S\backslash\{i\}} q_j(S)\, \phi_i(S\backslash\{j\}). && \text{(D.6)}
\end{aligned}$$

We want to show that $\phi(N)$ defined by the recursion of Equation D.6 is a convex combination of points in the core. To do so, we generate a random ordering of the players via a random deletion process. Starting from a subset S with $|S| \geq 2$, choose $k \in S$ with probability $q_k(S)$. Replace S with $S\backslash\{k\}$ and repeat this process until one player remains, at which point it stops. Associate with this deletion process the permutation:

$$\sigma^S = (i_1 \ldots, i_{|S|}), \tag{D.7}$$

obtained by recording the players in reverse deletion order, so that $i_{|S|}$ is the first player deleted and i_1 is the final remaining player. Write the usual marginal contribution of player i in the permutation σ as:

$$\mathsf{m}_i(\sigma) = v\big(P_{\sigma^S,i} \cup \{i\}\big) - v\big(P_{\sigma^S,i}\big), \tag{D.8}$$

where $P_{\sigma^S,i}$ is the set of predecessors of i in σ^S. Define, for each subset S and each i in S:

$$\psi_i(S) = \mathrm{E}\big[\mathsf{m}_i(\sigma^S)|S\big]. \tag{D.9}$$

For a convex game, every marginal-contribution vector $\mathsf{m}(\sigma^N)$ lies in the core (Shapley 48, 1971; Weber 57, 1988). Since $\psi(N)$ is an expectation, and therefore a convex combination, of these marginal-contribution vectors, it follows that $\psi(N)$ also lies in the core.

It remains to show that $\psi_i(S) = \phi_i(S)$, which we do by induction on $|S|$. The base case is immediate from $\psi_i(\{i\}) = v(\{i\}) = \phi_i(\{i\})$. Next, let $|S| \geq 2$ and suppose the result holds for all strict subsets of S. Conditioning on the first player deleted gives:

$$\psi_i(S) = \sum_{j \in S} q_j(S)\, \mathrm{E}\big[\mathsf{m}_i(\sigma^S) \,|\, i_{|S|} = j\big]. \tag{D.10}$$

If $j = i$, then i is the first player deleted and therefore the last player in the reverse deletion order. Its set of predecessors is then $S\backslash\{i\}$, and so:

$$\mathrm{E}\big[\mathsf{m}_i(\sigma^S) \,|\, i_{|S|} = j\big] = v(S) - v(S\backslash\{i\}). \tag{D.11}$$

If $j \neq i$, then, conditional on j being deleted first, the remaining deletion process is the same as the process starting from $S\backslash\{j\}$. Also, we know that j appears after i in the reverse deletion order and is not a predecessor of i. Therefore:

$$\mathrm{E}\big[\mathsf{m}_i(\sigma^S) \,|\, i_{|S|} = j\big] = \begin{cases} v(S) - v(S\backslash\{i\}) & \text{if } j = i, \\ \phi_i(S\backslash\{j\}) & \text{if } j \neq i. \end{cases} \tag{D.12}$$

Substituting this into Equation D.10 yields:

$$\psi_i(S) = q_i(S)\, \big[v(S) - v(S\backslash\{i\})\big] + \sum_{j \in S\backslash\{i\}} q_j(S)\, \phi_i(S\backslash\{j\}), \tag{D.13}$$

which is exactly the recursion of Equation D.6, completing the proof. □

Appendix E: Proofs for Section 8

We start by observing that the hitting-probability formulation of the CCV (Proposition 2) continues to hold when the breakdown weights are signed.

Proof of Theorem 5. We employ unanimity games to establish the result. Fix nonempty subsets $T \subseteq S \subseteq N$. If $T = S$, then $\lambda_T(T) = 1$ by definition. Suppose that $T \subsetneq S$ and consider the unanimity game u_T. Fix i in T. By hypothesis, the CCV coincides with the Shapley value for every characteristic function, so:

$$\phi_i(S) = \phi_i^{\mathrm{Sh}}(S) = \frac{1}{|T|}. \tag{E.1}$$

Write the total signed weight of partitions of S in which T remains intact in the cell containing i:

$$q_S(T) = \sum_{\{\pi \in \Pi(S) \,:\, T \subseteq \pi_i\}} p_S(\pi). \tag{E.2}$$

In the unanimity game u_T, if a partition π splits T, then the continuation game on the cell containing i is identically zero, and therefore $\phi_i(\pi_i) = 0$. If π leaves T intact, then $T \subseteq \pi_i \subsetneq S$, and by the hypothesis that the CCV equals the Shapley value, we have $\phi_i(\pi_i) = \dfrac{1}{|T|}$. Therefore, the CCV recursion gives:

$$\phi_i(S) = \frac{m(S)}{|S|} + \frac{q_S(T)}{|T|}. \tag{E.3}$$

For the same unanimity game, the local loss at S is 0 when T survives intact and 1 when T is split. Using the fact that the signed breakdown weights sum to 1, this implies $m(S) = 1 - q_S(T)$, so that:

$$\phi_i(S) = \frac{1 - q_S(T)}{|S|} + \frac{q_S(T)}{|T|}. \tag{E.4}$$

But $\phi_i(S) = 1/|T|$, from which

$$\left(\frac{1}{|S|} - \frac{1}{|T|}\right)(1 - q_S(T)) = 0, \tag{E.5}$$

so that, using $|T| \neq |S|$, we obtain $q_S(T) = 1$.

The same argument applies to every strict superset $U \supsetneq T$, so $q_U(T) = 1$ for all such U. This establishes that, in the unanimity game u_T, we have $m(U) = 0$ for every $U \supsetneq T$. If $T \not\subseteq U$, then u_T is identically 0 on U and all of its subsets, so again $m(U) = 0$. At T, we have $m(T) = u_T(T) = 1$.

It follows that, in the hitting-probability representation of the CCV at S, the only nonzero term involving player i in T is the term corresponding to T:

$$\phi_i(S) = \lambda_S(T)\,\frac{m(T)}{|T|} = \frac{\lambda_S(T)}{|T|}. \tag{E.6}$$

Since $\phi_i(S) = 1/|T|$ and $m(T) = 1$, it follows that $\lambda_S(T) = 1$. Because $T \subseteq S$ was arbitrary, we conclude that $\lambda_S(T) = 1$, as required. □

Proposition E.1. *The 1BB weights sum to* 1.

Proof. Let $n = |S|$. For $1 \leq k \leq n-2$, there are $\binom{n}{k}$ 1BB partitions in which exactly k players are singletons, and each has weight $(-1)^{k+1}$. The discrete partition has weight $(-1)^n(n-1)$. Therefore:

$$\sum_{\pi \in \Pi(S)} p_S^{\mathrm{1BB}}(\pi) = \sum_{k=1}^{n-2} \binom{n}{k} (-1)^{k+1} + (-1)^n(n-1). \tag{E.7}$$

Using:

$$0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k, \tag{E.8}$$

and separating the terms $k = 0, n-1, n$, we get:

$$\sum_{k=1}^{n-2} \binom{n}{k} (-1)^{k+1} = 1 - (-1)^n(n-1). \tag{E.9}$$

Substitution yields:

$$\sum_{\pi \in \Pi(S)} p_S^{\mathrm{1BB}}(\pi) = 1, \tag{E.10}$$

as required. □

Proof of Proposition 5. Fix a nonempty subset $T \subseteq S \subseteq N$. If $T = S$, then $\lambda_S(T) = 1$ by definition. Suppose first that $2 \leq |T| < |S|$. Under the 1BB weights, T can survive a breakdown of S only if it is contained in the unique nonsingleton block. If exactly k players in $S\backslash T$ are separated as singletons, where $1 \leq k \leq |S\backslash T|$, the number of such partitions is:

$$\binom{|S\backslash T|}{k}, \tag{E.11}$$

each with weight $(-1)^{k+1}$. The total signed weight of partitions in which T survives intact is therefore:

$$\sum_{k=1}^{|S\backslash T|} \binom{|S\backslash T|}{k} (-1)^{k+1} = 1, \tag{E.12}$$

using the alternating-binomial identity.

We now prove $\lambda_S(T) = 1$ by induction on $|S\backslash T|$. The result is immediate when $|S\backslash T| = 0$. Suppose it is true whenever the size difference is smaller than $|S\backslash T|$. Conditioning on the first breakdown of S:

$$\lambda_S(T) = \sum_{\{\pi \,:\, T \subseteq B_\pi\}} p_S^{\mathrm{1BB}}(\pi)\, \lambda_{B_\pi}(T), \tag{E.13}$$

where B_π denotes the unique nonsingleton block of π. For every such partition, we have $T \subseteq B_\pi \subsetneq S$, so the induction hypothesis gives $\lambda_{B_\pi}(T) = 1$, and therefore:

$$\lambda_S(T) = \sum_{\{\pi \,:\, T \subseteq B_\pi\}} p_S^{\mathrm{1BB}}(\pi) = 1. \tag{E.14}$$

It remains to consider $|T| = 1$. Write $T = \{i\}$. Every partition of S contains a unique cell containing i, and, conditional on that first breakdown, the continuation hitting weight of $\{i\}$ is 1 by induction on $|S|$. Therefore:

$$\lambda_S(\{i\}) = \sum_{\pi \in \Pi(S)} p_S^{\mathrm{1BB}}(\pi) = 1, \tag{E.15}$$

where the last equality follows from Proposition E.1. This establishes the result. □

To prove Proposition 6, we first establish two lemmas characterizing the expected local losses and hitting probabilities under the lower-interval Möbius distribution.

Lemma E.1. *Under the lower-interval Möbius distribution, we have:*

$$m(T) = v(T) - \frac{1}{|T| - 1} \sum_{j \in T} v(T\backslash\{j\}), \tag{E.16}$$

for every subset T of N with $|T| \geq 2$.

Proof. Fix $T \subseteq N$ with $|T| \geq 2$. By definition:

$$m(T) = v(T) - \sum_{\pi \in \Pi(T)} p_T^{\mathrm{mo}}(\pi) \sum_{U \in \pi} v(U). \tag{E.17}$$

We collect the coefficients on each term $v(W)$ for $W \subsetneq T$. First suppose $W = T\backslash\{j\}$ for some j in T. The only partition of T containing W as a cell is $\{T\backslash\{j\}, \{j\}\}$. Its lower Möbius weight is:

$$p_T^{\mathrm{mo}}\big(\{T\backslash\{j\}, \{j\}\}\big) = \frac{(-1)^2(|T| - 2)!\, 0!}{(|T| - 1)!} = \frac{1}{|T| - 1}. \tag{E.18}$$

Therefore, terms of this form contribute the following to the expected continuation value in Equation E.17:

$$\frac{1}{|T|-1}\sum_{j\in T} v(T\backslash\{j\}). \tag{E.19}$$

Next let $W \subsetneq T$ satisfy $|T\backslash W| \geq 2$. The coefficient on $v(W)$ is the sum of the lower Möbius weights over all partitions of T containing W as a cell, namely:

$$\sum_{W\in\pi\in\Pi(T)} p_T^{\mathrm{mo}}(\pi). \tag{E.20}$$

These partitions are in one-to-one correspondence with all partitions ρ of $T\backslash W$, including the trivial partition, via $\rho \mapsto \rho \cup \{W\}$. By the Möbius cancellation identity on the partition lattice, this sum is 0 whenever $|T\backslash W| \geq 2$. Therefore all terms $v(W)$ with $|T\backslash W| \geq 2$ cancel, leaving only the terms of the form of Expression E.19, as required. □

Lemma E.2. *Under the lower-interval Möbius distribution, we have:*

$$\lambda_S(T) = \binom{|S|-1}{|T|-1}^{-1}, \tag{E.21}$$

for every $T \subseteq S \subseteq N$ with $|T| \geq 2$.

Proof. We first calculate the total weight of breakdowns of a set $U \supsetneq T$ that can contribute to the hitting probability of T. As in the proof of Lemma E.1, the total lower Möbius weight of partitions in which the cell containing T has complement of size at least 2 is 0. Therefore, as before, only partitions of the form $\{U\backslash\{j\}, \{j\}\}$, for $j \in U\backslash T$, contribute to the hitting probability of T. For each such j, the weight is given by Equation E.18 with U in place of T. There are $|U\backslash T|$ choices of $j \in U\backslash T$. Under the lower Möbius distribution, the continuation hitting probability after any such exit depends only on the size of the remaining set, so that:

$$\lambda_U(T) = \frac{|U\backslash T|}{|U|-1}\,\lambda_{U\backslash\{j\}}(T), \tag{E.22}$$

for any $j \in U\backslash T$. Starting from $\lambda_T(T) = 1$ and iterating this recursion from T up to S gives:

$$\lambda_S(T) = \prod_{u=|T|+1}^{|S|} \frac{u-|T|}{u-1}, \tag{E.23}$$

so that:

$$\lambda_S(T) \;= \frac{1\times 2\times\cdots\times|S\backslash T|}{|T|(|T|+1)\cdots(|S|-1)} = \frac{(|S\backslash T|!\,(|T|-1)!}{(|S|-1)!}, \tag{E.24}$$

which is Equation E.21. □

Proof of Proposition 6. We use the hitting-probability representation of the CCV. For $T = \{i\}$, we have $m(\{i\}) = v(\{i\})$ and $\lambda_N(\{i\}) = 1$, so that the singleton term is $v(\{i\})$. Now fix T with $|T| \geq 2$. Combining Equations E.16 and E.21, we obtain:

$$\phi_i^{\mathrm{mo}}(N) = v(\{i\}) + \sum_{\substack{i\in T\subseteq N\\ |T|\geq 2}} \binom{|N|-1}{|T|-1}^{-1} \frac{1}{|T|}\big[v(T) - \frac{1}{|T|-1}\sum_{j\in T} v(T\backslash\{j\})\big], \tag{E.25}$$

which is Equation 95. □

Appendix F: Calculations for Example 13

We first formally justify the decomposition used in the main text. Fix a contextual subset-refinement graph and breakdown weights. Let u and w be two contextual characteristic functions and set:

$$v = \alpha u + \beta w, \tag{F.1}$$

context by context. For every context (S, r_S) and every $\pi \in \Pi(S)$:

$$l_\pi^v(S, r_S) = \alpha\, l_\pi^u(S, r_S) + \beta\, l_\pi^w(S, r_S), \tag{F.2}$$

so that:

$$m^v(S, r_S) = \alpha\, m^u(S, r_S) + \beta\, m^w(S, r_S). \tag{F.3}$$

Induction on the size of S in the extended CCV recursion of Definition 12 then gives:

$$\phi_i^v(S, r_S) = \alpha\, \phi_i^u(S, r_S) + \beta\, \phi_i^w(S, r_S) \tag{F.4}$$

at every context. This establishes that the extended CCV is additive in the contextual characteristic function. Note that the argument is valid for both unsigned and signed breakdown weights.

For Example 13, u assigns value 1 to the grand coalition and 0 to every other context, and w assigns value 1 to player a_1 when all other players are singletons and 0 to every other context. For the component u, all continuation payoffs below the top are 0. Therefore, every breakdown of the grand coalition has a local loss of 1 and the expected local loss is 1. Writing $n = |N|$, this yields, for every player i:

$$\phi_i^u(N) = \frac{1}{n}. \tag{F.5}$$

It remains to calculate the payoff generated by w.

Definition F.1. *A context (S, r_S) with $a_1 \in S$ is **relevant** if every cell previously separated from the path to S is a singleton.*

At any non-relevant context (S, r_S), the partial run r_S includes at least one previously separated non-singleton cell. That cell is not refined further on any continuation that repeatedly follows the cell containing a_1. It follows that w assigns value 0 to (S, r_S) and all such continuation contexts. Every other cell in an intervening breakdown does not contain a_1 and is also assigned value 0 by w. Therefore, the extended CCV gives a_1 a payoff of 0 at every non-relevant context.

Because w is path-independent and the breakdown weights for the upper Möbius, 1BB, and lower Möbius distributions are exchangeable, we can identify relevant contexts whose cells containing a_1 are of the same size. Let x_s denote this common payoff to a_1 at a cell of size s. Then $x_1 = 1$. At a relevant context of size $s \geq 2$, the only breakdown with a nonzero local loss is complete breakdown into singletons, and the local loss is then -1. Also, a continuation payoff can be nonzero only when a_1 remains in a cell of some size $t < s$ and every other player becomes a singleton. Using $w(N) = 0$, Efficiency gives:

$$\sum_{i \in N} \phi_i^w(N) = 0, \tag{F.6}$$

so that we can write:

$$\phi_{a_i}^v(N) = \begin{cases} \dfrac{1}{n} + x_n & \text{for } i = 1, \\ \dfrac{1}{n} - \dfrac{x_n}{n-1} & \text{for } i = 2, \ldots, n. \end{cases} \tag{F.7}$$

Proposition F.1. *For each $s \geq 2$, the upper Möbius distribution gives:*

$$x_s = \frac{(-1)^s (s-2)!}{s}. \tag{F.8}$$

Proof. Suppose the current cell has size s and a breakdown leaves a_1 in a cell of size $t < s$, with every other player becoming a singleton. The number of such partitions is:

$$\binom{s-1}{t-1}, \tag{F.9}$$

each with weight $(-1)^{s-t+1}(s-t)!$. For $t = 1$, this is the complete breakdown, whose weight is $(-1)^s(s-1)!$. Since the local loss is -1, the extended CCV gives:

$$x_s = \frac{(-1)^{s+1}(s-1)!}{s} + \sum_{t=1}^{s-1} \binom{s-1}{t-1} (-1)^{s-t+1}(s-t)!\, x_t. \tag{F.10}$$

From this, we find $x_2 = 1/2$ as required. Suppose inductively that Equation F.8 holds for $2 \leq t < s$.

Separating the $t = 1$ term and substituting the induction hypothesis gives:

$$x_s = (-1)^s(s-1)!\big[1 - \frac{1}{s} - \sum_{t=2}^{s-1} \frac{1}{t(t-1)}\big] = (-1)^s(s-1)!\big[\frac{1}{s-1} - \frac{1}{s}\big] \tag{F.11}$$

$$= \frac{(-1)^s(s-2)!}{s}, \tag{F.12}$$

where we used:

$$\sum_{t=2}^{s-1} \frac{1}{t(t-1)} = 1 - \frac{1}{s-1}. \tag{F.13}$$

This completes the proof. □

Proposition F.2. *The 1BB distribution gives:*

$$x_2 = \frac{1}{2},\ x_s = -\frac{1}{s} \text{ for } s \geq 3. \tag{F.14}$$

Proof. A breakdown that leaves a_1 in a nonsingleton cell of size t, with the other $s - t$ players becoming singletons, has weight $(-1)^{s-t+1}$. Complete breakdown has weight $(-1)^s(s-1)$. Therefore, the weighted local loss contribution from complete breakdown is $(-1)^{s+1}(s-1)$, while the continuation contribution is $(-1)^s(s-1)x_1$. From this:

$$x_s = \frac{(-1)^{s+1}(s-1)}{s} + (-1)^s(s-1) + \sum_{t=2}^{s-1} \binom{s-1}{t-1} (-1)^{s-t+1}\, x_t \tag{F.15}$$

$$= \frac{(-1)^s(s-1)^2}{s} + \sum_{t=2}^{s-1} \binom{s-1}{t-1} (-1)^{s-t+1}\, x_t. \tag{F.16}$$

Direct calculation gives $x_2 = 1/2$. Suppose inductively that Equation F.14 holds for $2 \leq t < s$. Using:

$$\frac{1}{t}\binom{s-1}{t-1} = \frac{1}{s}\binom{s}{t}, \tag{F.17}$$

we obtain:

$$\sum_{t=2}^{s-1} \binom{s-1}{t-1} (-1)^{s-t+1}\, x_t = \frac{1}{s} \sum_{t=2}^{s-1} (-1)^{s-t} \binom{s}{t} + (-1)^{s-1}(s-1) \tag{F.18}$$

$$= -\frac{(-1)^s(s-1)^2 + 1}{s}, \tag{F.19}$$

where the second term in the middle equality corrects for $x_2 = 1/2$ rather than $-1/2$, and the final equality uses the alternating-binomial identity. Putting Equations F.16 and F.19 together establishes the result. □

Proposition F.3. *For each $s \geq 2$, the lower Möbius distribution gives:*

$$x_s = \frac{1}{s!}. \tag{F.20}$$

Proof. Each relevant partition that leaves a_1 in a cell of size $t < s$ has weight:

$$(-1)^{s-t+1}\, \frac{(t-1)!}{(s-1)!}. \tag{F.21}$$

The number of such partitions is:

$$\binom{s-1}{t-1}, \tag{F.22}$$

so their total weight is:

$$\binom{s-1}{t-1}(-1)^{s-t+1}\, \frac{(t-1)!}{(s-1)!} = \frac{(-1)^{s-t+1}}{(s-t)!}. \tag{F.23}$$

This formula also applies to $t = 1$, corresponding to complete breakdown. Therefore, the extended CCV is:

$$x_s = \frac{(-1)^{s+1}}{s!} + \sum_{t=1}^{s-1} \frac{(-1)^{s-t+1}}{(s-t)!}\, x_t. \tag{F.24}$$

From this, we find $x_2 = 1/2$ as required. Suppose inductively that Equation F.20 holds for $2 \leq t < s$. Then, using the alternating-binomial identity, Equation F.24 becomes:

$$x_s = \frac{1}{s!}\left[(-1)^{s+1} + \sum_{t=1}^{s-1}(-1)^{s-t+1}\binom{s}{t}\right] = \frac{1}{s!}, \tag{F.25}$$

completing the proof. □

Using Propositions F.1, F.2, and F.3 in Equation F.6, and adding $\phi^u(N)$ from Equation F.5, we obtain the corresponding entries in Table 1 in Section 9.2.

Appendix G: Proofs for Section 10

Proof of Proposition 9. The idea of the proof is to separate each player's original payoff into the amount accumulated before the first nontrivial projected partition and the continuation payoff after that. Induction identifies the latter with the reduced continuation payoff, while the former is common to all members of T and is therefore recovered by equal division of the reduced local loss.

Formally, we proceed by induction on $|T|$. If $T = \{j\}$, the result is immediate from the definition of $v^{-i,\phi}(T,\tau)$. Now suppose $|T| \geq 2$ and the result is true at every reduced context whose subset is a strict subset of T. Fix (T,τ) and, to simplify notation, write:

$$Y_j = \mathbb{E}_q\big[\phi_j(C^r(T,\tau)) \;\big|\; H_{T,\tau}\big], \tag{G.1}$$

and, for player j's continuation payoff in the reduced game:

$$V_j = \sum_{\pi \in \Pi(T)} p^{-i}_{T,\tau}(\pi)\, \phi^{-i}_j(\pi_j, \tau_{\pi,\pi_j}). \tag{G.2}$$

By the induction hypothesis, the definition of $p^{-i}_{T,\tau}$, and the law of iterated expectations, V_j is the conditional expected original continuation payoff to player j following the first nontrivial projected partition.

Next, apply the original CCV recursion successively, starting at $C^r(T,\tau)$ and continuing until that partition is reached. Until then, all players in T remain together, so the conditional expectation of their accumulated shares of local losses is the same for every player in T. Denote this common amount by $W(T,\tau)$. It follows that, for all j in T:

$$Y_j = W(T,\tau) + V_j. \tag{G.3}$$

By the definition of the reduced characteristic function:

$$v^{-i,\phi}(T,\tau) = \sum_{j \in T} Y_j. \tag{G.4}$$

Efficiency at the immediately succeeding reduced contexts gives:

$$\sum_{\pi \in \Pi(T)} p^{-i}_{T,\tau}(\pi) \sum_{U \in \pi} v^{-i,\phi}(U, \tau_{\pi,U}) = \sum_{j \in T} V_j. \tag{G.5}$$

Therefore, the reduced expected local loss is:

$$m^{-i}(T,\tau) = v^{-i,\phi}(T,\tau) - \sum_{\pi \in \Pi(T)} p^{-i}_{T,\tau}(\pi) \sum_{U \in \pi} v^{-i,\phi}(U, \tau_{\pi,U}) \tag{G.6}$$

$$= \sum_{j \in T}(Y_j - V_j) = |T| \times W(T,\tau). \tag{G.7}$$

The reduced CCV recursion now yields:

$$\phi^{-i}_j(T,\tau) = \frac{m^{-i}(T,\tau)}{|T|} + V_j = W(T,\tau) + V_j = Y_j, \tag{G.8}$$

as required. □

Proof of Proposition 10. Suppose the CCV coincides with the Shapley value. Then, by Theorem A in Hart and Mas-Colell (14, 1989), the CCV has a potential representation. Any (discrete) gradient field is curl-free, since the circulation around every elementary square is equal to 0. Therefore, the CCV is curl-free.

Conversely assume that the CCV is curl-free. By the discrete analog to the Poincaré lemma of vector calculus, there is a function $P : 2^N \to \mathbb{R}$, unique up to an additive constant, such that, for all nonempty $S \subseteq N$ and $i \in S$:

$$\phi_i(S) = P(S) - P(S\backslash\{i\}). \tag{G.9}$$

Fix the normalization $P(\emptyset) = 0$. Summing Equation G.9 over i in S and using Efficiency on every subset, we obtain, for every nonempty $S \subseteq N$:

$$|S| \cdot P(S) - \sum_{i \in S} P(S\backslash\{i\}) = v(S). \tag{G.10}$$

This is the discrete Poisson equation on the subset lattice. Theorem A in Hart and Mas-Colell (14, 1989) establishes that the solution is unique and has gradient equal to the Shapley value on every subset. We conclude that the CCV coincides with the Shapley value, as required. □

The proof of Theorem 8 follows an analogous strategy to that in Young (58, 1985), using local expected losses in place of unanimity games as a basis.

Lemma G.1. *Fix (N, G, p). For any collection of numbers:*

$$\{\mu(S) : \emptyset \neq S \subseteq N\}, \tag{G.11}$$

there is a unique characteristic function v with $m^v(S) = \mu(S)$ for every nonempty subset S.

Proof. For a singleton $\{i\}$, use $m^v(\{i\}) = v(\{i\})$ to set $v(\{i\}) = \mu(\{i\})$. Now suppose inductively that $v(T)$ has been defined for every T with $|T| < |S|$. By definition:

$$m^v(S) = v(S) - \sum_{\pi \in \Pi(S)} p_S(\pi) \sum_{T \in \pi} v(T), \tag{G.12}$$

and every set appearing in the second term is a strict subset of S, so that term is already determined. Therefore, there is a unique choice:

$$v(S) = \mu(S) + \sum_{\pi \in \Pi(S)} p_S(\pi) \sum_{T \in \pi} v(T), \tag{G.13}$$

that achieves $m_v(S) = \mu(S)$. Proceeding upwards constructs a unique v with the given collection of expected local losses. □

Note that the hitting probabilities are fixed by (N, G, p). So, Lemma G.1 says that we can independently vary the CCV dividends:

$$d^v(S) = \lambda_N(S)\, m^v(S), \tag{G.14}$$

whenever $\lambda_N(S) \neq 0$. Also, Theorem 8 concerns noncontextual subset-refinement games, which enabled us to define $m^v(S)$ independently of the occurrence of S in the graph.

Proof of Theorem 8. First note that Strong Monotonicity, applied in both directions, implies that if $d^v(U) = d^w(U)$ for every $U \ni i$, then $\psi_i^v = \psi_i^w$. We prove uniqueness by induction on the number of nonzero dividends $d^v(S)$. If all dividends are 0, Strong Monotonicity in both directions implies that every player's payoff is equal to its payoff in the game identically equal to 0. Efficiency and Symmetry give $\psi_i^v = 0$ for every i. This agrees with the CCV.

Now suppose the result holds whenever there are at most k nonzero dividends, and let v have $k+1$ nonzero dividends. Consider a player i who does not belong to every subset with nonzero dividend (if such a player exists). Choose S^* so that $i \notin S^*$ and $d^v(S^*) \neq 0$. By Lemma G.1, there is a characteristic function w whose dividends agree with those of v except that $d^w(S^*) = 0$. This says that w has only k nonzero dividends. Since $i \notin S^*$, all dividends on sets containing i are unchanged, from which $\psi_i^v = \psi_i^w$. By the induction hypothesis, we have $\psi_i^w = \phi_i^w$. Also, from the CCV formula:

$$\phi_i^v = \sum_{i \in U \subseteq N} \frac{d^v(U)}{|U|}, \tag{G.15}$$

the change in $d^v(S^*)$ does not affect player i. We therefore get $\phi_i^w = \phi_i^v$, and so $\psi_i^v = \phi_i^v$.

This argument determines the payoff to every player who is absent from at least one set with nonzero dividend. Any remaining players belong to every such set. That is, any subset containing one but not another such player has dividend 0. By Efficiency, the total payoff left after the already determined players are paid is fixed. By Symmetry, the remaining players must divide this residual equally. The CCV does the same, since every nonzero dividend contains all of the remaining players. Therefore, their payoffs coincide with their CCV payoffs. This completes the induction and establishes uniqueness.

Conversely, the CCV satisfies Efficiency and Symmetry, and Strong Monotonicity follows immediately from the hitting formula for the CCV. □

Proof of Proposition 11. We proceed by induction on the size of S. The result is immediate when $S = \{i\}$, since player i receives $v(\{i\}) = \phi_i(\{i\})$. Now fix a nonsingleton subset S and suppose the result holds for every strict subset of S. Following rejection at S, player i's expected continuation payoff, independent of who rejects, is:

$$\sigma_i(S) = \sum_{\pi \in \Pi(S)} p_S(\pi)\, \phi_i(\pi_i), \tag{G.16}$$

since, conditional on partition π, the induction hypothesis gives player i the continuation payoff $\phi_i(\pi_i)$ in the cell containing i. By Efficiency of the CCV at every strict subset of S:

$$\sum_{i \in S} \sigma_i(S) = \sum_{\pi \in \Pi(S)} p_S(\pi) \sum_{i \in S} \phi_i(\pi_i) \tag{G.17}$$

$$= \sum_{\pi \in \Pi(S)} p_S(\pi) \sum_{T \in \pi} v(T) \tag{G.18}$$

$$= v(S) - m(S). \tag{G.19}$$

Suppose player k is selected as proposer. If the proposal is rejected, each responder $i \neq k$ receives the continuation payoff $\sigma_i(S)$. By backward induction through the ordered responses and the tie-breaking assumption, a proposal is therefore accepted if and only if every responder $i \neq k$ receives at least $\sigma_i(S)$. Therefore, conditional on reaching agreement, the proposer maximizes its payoff by offering each responder exactly $\sigma_i(S)$. The proposer then retains

$$v(S) - \sum_{i \in S \setminus \{k\}} \sigma_i(S) = v(S) - \sum_{i \in S} \sigma_i(S) + \sigma_k(S) = m(S) + \sigma_k(S), \tag{G.20}$$

where the final equality follows from Equation G.19. Since $m(S) \geq 0$ by assumption, this payoff is at least the proposer's continuation payoff $\sigma_k(S)$. Therefore, when $m(S) > 0$, the proposer strictly prefers to make an acceptable proposal, while when $m(S) = 0$, agreement is selected by the tie-breaking assumption.

Player i is selected as proposer with probability $1/|S|$. Therefore, its expected payoff at S is

$$\frac{1}{|S|}\big(\sigma_i(S) + m(S)\big) + \frac{|S| - 1}{|S|}\sigma_i(S) = \sigma_i(S) + \frac{m(S)}{|S|} = \phi_i(S), \tag{G.21}$$

where the final equality is the CCV recursion. It follows that, assuming the result for every strict subset of S, the unique backward-induction payoff in the continuation game beginning at S is the CCV payoff vector $\phi(S)$. By induction, this holds for every nonempty subset S. Setting $S = N$ proves the result. □